Santiago Huerta
E. T.S. de Arquitectura
Universidad Politécnica de Madrid
Avda. Juan de Herrera, 4
28040 Madrid SPAIN
Santiago.Huerta@upm.es

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Research

Galileo was Wrong: The Geometrical Design of Masonry Arches

Abstract. Since antiquity master builders have always used simple geometrical rules for designing arches. Typically, for a certain form, the thickness is a fraction of the span. This is a proportional design independent of the scale: the same ratio thickness/span applies for spans of 10m or 100m. Rules of the same kind were also used for more complex problems, such as the design of a buttress for a cross-vault. Galileo attacked this kind of proportional design in his *Dialogues*. He stated the so-called square-cube law: internal stresses grow linearly with scale and therefore the elements of the structures must become thicker in proportion. This law has been accepted many times uncritically by historians of engineering, who have considered the traditional geometrical design as unscientific and incorrect. In fact, Galileo's law applies only to strength problems. Stability problems, such as the masonry arch problem, are governed by geometry. Therefore, Galileo was wrong in applying his reasoning to masonry buildings

Introduction

Arches are the essential element of masonry construction. They were invented some 6,000 years ago in Mesopotamia. The first arches were small; they were used to cover tombs. It is fascinating to look at the structural experimentation which developed over the course of 2,000 years before the arches emerged from the earth and began to form part of architecture proper [Besenval 1984; El-Naggar 1999]. The barns of the Ramesseum (thirteenth century BC) were covered by barrel vaults and the Hanging Gardens of Babylon (seventh century BC) where, in fact, supported on a system of arches and vaults. In Europe the Etruscans where among the first to use stone arches of moderate size, but it was in Imperial Rome, when the arch and the vault began to form an essential part of architecture, that the spans grew in order of magnitude, reaching the 43m of the Pantheon. Since then and up to the nineteenth century arches and vaults were at the heart of structural design: "All the aspects of architecture are derived from the vault", said Auguste Choisy, and the history of the different architectonic styles is also the history of how to solve the technical problem of building arches and vaults of different forms in brick, stone and mortar.

An arch thrusts: the stones, trying to fall down due to the force of gravity, produce inclined forces which are transmitted within the masonry down the springings (fig. 1). The forces must be inclined to give vertical equilibrium, but at

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the same time they produce a horizontal component. The vertical loads increase from top (the keystone) to the springings, but the horizontal component remains constant all through the arch to maintain horizontal equilibrium. At the springings of the arch there is, then, an inclined force which must be resisted by the *buttresses*. The structural design of masonry architecture deals with two fundamental problems: 1) To design arches which will stand; 2) To build buttresses which will withstand their thrust. Roman solutions are very different from Gothic or Baroque solutions, but the problem remains the same: to obtain a safe state of equilibrium which will guarantee the life of the building for centuries or millennia.

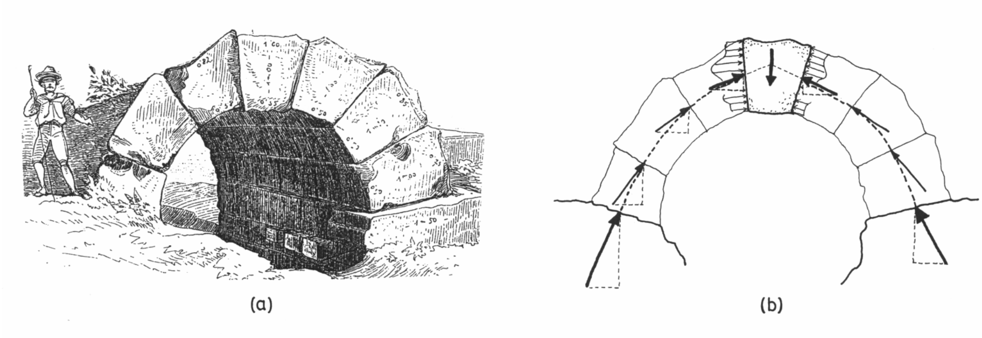


Fig. 1. (a) Etruscan voussoir arch [Durm 1885]; (b) Equilibrium of the stones in an arch. The inclined forces are transmitted within the arch and at the abutments there is always a thrust with a horizontal component (uniform through the entire arch) which must be resisted

The question is, how were arches and buttresses designed? The scientific theory of structures was applied only during the nineteenth century and this fact leaves almost all historical architecture outside the realm of this branch of modern applied mechanics. However, it is evident that the great buildings of the past could not have been built without some kind of knowledge: the master builders used a theory of a different kind, based in the critical observation of masonry building processes. This "non-scientific" theory must have been rich and complex, because its application resulted in the Pantheon, the Gothic cathedrals and the Hagia Sophia.

Proportional design of arches, vaults and buttresses. We know that this traditional theory, or the particular expression of it in every particular epoch, was condensed in the form of structural rules. For example, to design an arch of a certain form, the relevant parameter is the thickness, and this was always obtained as a fraction of the span. The same occurs with the buttresses, whose depth was calculated, again, as fraction of the span. Of course, the rules were specific for each structural type: the Gothic rules for buttresses give the depth of the buttresses as nearly 1/4 of the span; Renaissance rules give between 1/3 and 1/2 of the span. Gothic vaults are much lighter than Renaissance vaults, but the approach is the same.

The beginning of one of the manuscripts which have survived from the late Gothic period (*Vom des Chores Mass* (The measure of the Choir) published in [Coenen 1990]) expresses the method in a clear way:

> The building follows precise laws and all its parts are ruled, in such a way that all its elements are related with the whole building and the whole building is related with each one of its parts. The Choir is the fundament and the origin of all the rules, and from its span we obtain not only the thickness of its wall, but also the templates of the imposts and of all the elements of the work.

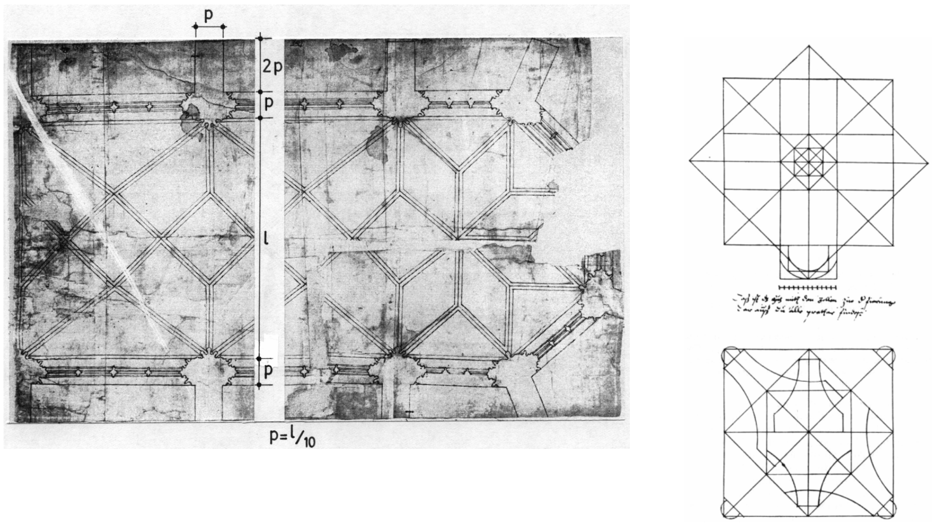


Fig. 2. Late Gothic structural design of a church and its elements. *Left,* the span is the "great module" from which all the elements are derived (letters superposed on an original Gothic drawing from [Koepf 1969]). *Right*, the wall thickness (span/10) is the "little module" from which the ribs, mullions and imposts are obtained, employing the *ad quadratum* technique of rotating squares [Hecht 1979].

The span of the vault of the choir, let us call it s, is a "great module" and all the dimensions of the structural elements are obtained as fractions of it. The wall should be $s/10$ and the buttresses three times this quantity, i.e., $3s/10$. The templates of the ribs and the mullions of the windows were also obtained from the wall thickness (fig. 2).

There were other rules for buttress design as well which have survived through the successive copies and re-elaborations of the medieval stone-cutting manuals, part of which were incorporated in Renaissance and Baroque stone-cutting handbooks. In fig. 3 a geometrical rule is represented. The intrados of the transverse arch of the vault is divided in three parts and a line is traced joining one of the points with the springings. Then the same distance is taken as the prolongation of the line and this point gives the depth of the buttress.

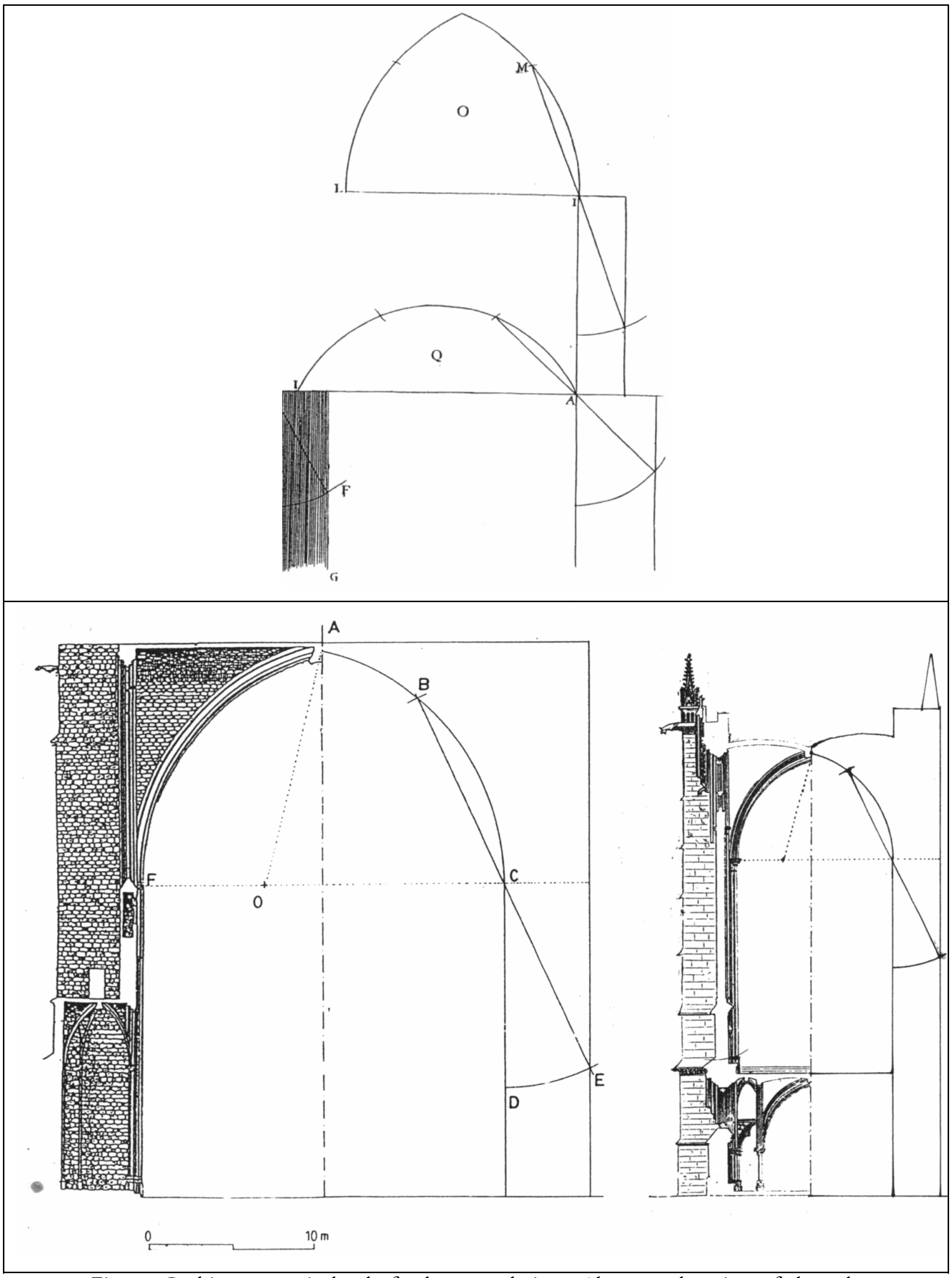


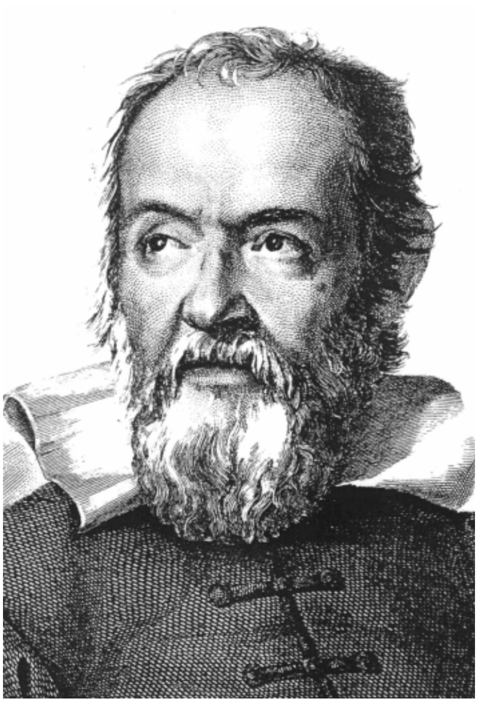
Fig. 3. Gothic geometrical rule for buttress design. *Above,* explanation of the rule [Derand 1643]. *Below*, application of the rule to two different buildings drawn to the same scale: the cathedral of Gerona and the Sainte Chapelle of Paris [Huerta 2004].

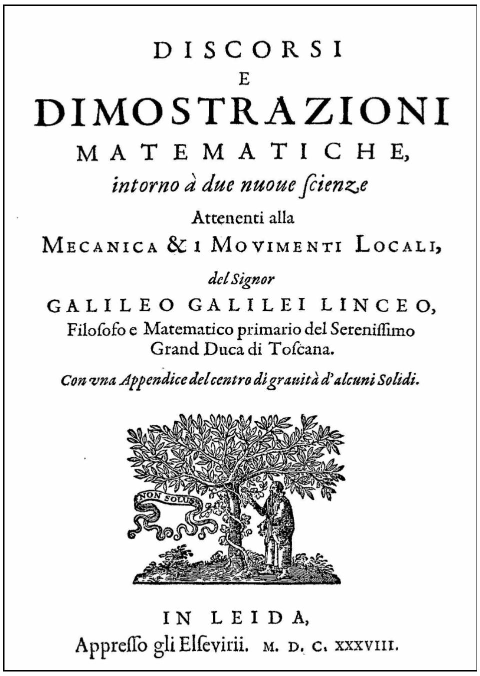
Rules of the same kind, arithmetical or geometrical, were employed in other periods as well – the Renaissance, Baroque, etc. – until in the nineteenth century masonry architecture began an accelerated decay, due to the appearance of new materials (iron, steel, reinforced concrete) and new structural types (frames, trusses, thin shells). However, structural design rules were also used all through the nineteenth century.

The essential characteristic of all these rules is that they are "proportional" and that they control the overall form of the structure of the building. It is a "geometrical design", which was considered to be correct for a building of any size.

Galileo and the "square-cube law"

Galileo (1564–1642) was the first to treat structural problems in a scientific way. During his forced reclusion in Arcetri, he wrote a book entitled *Discorsi e Dimostrazioni Matematiche intorno à due nuove sicenze Attenenti alla Mecanica & i movimenti Locali* (*Dialogues Concerning Two New Sciences*), published in 1638 (fig. 4). The two sciences were the Strength of Materials and the Cinematics, two topics less problematic than cosmology. It is the strength of materials which interests us.





DISCORSI
E
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MATEMATICHE,
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del Signor
GALILEO GALILEI LINCEO,
Filoſofo e Matematico primario del Sereniſſimo
Grand Duca di Toſcana.
Con vna Appendice del centro di grauità d'alcuni Solidi.

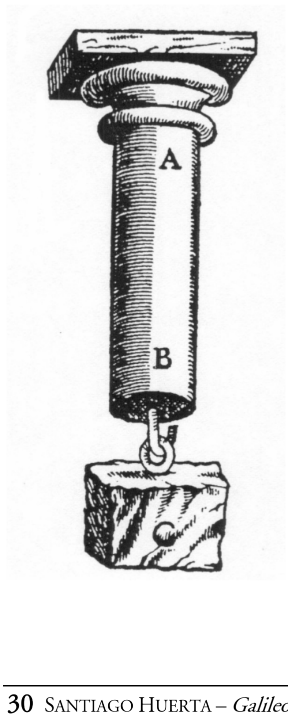
IN LEIDA,
Appreſſo gli Elſevirii. M. D. C. XXXVIII.

Fig. 4. *Left,* portrait of Galileo Galilei. *Right*, first page of the *Dialogues (Discorsi)* of 1638

Galileo was trying, for the first time, to draw scientific conclusions about the strength of beams, a problem of evident practical interest. However, just from the beginning he exposes the result of his researches and mounts an attack on medieval proportional design (this and the following quotations are from the 1954 translation of Crew and de Salvio):

> Therefore, Sagredo, you would do well to change the opinion which you, and perhaps also many other students of mechanics, have entertained concerning the ability of machines and structures to resist external disturbances, thinking that when they are built of the same material and maintain the same ratio between parts, they are able equally, or rather proportionally, to resist or yield to such external disturbances and blows. For we can demonstrate by geometry that the large machine is not proportionally stronger that the small. Finally we may say that, for every machine and structure, whether artificial or natural, there is set a necessary limit beyond which neither art nor nature can pass; it is here understood, of course, that the material is the same and the proportion preserved.

The strength of beams. Galileo, however, considers only the case of simple bending. It is obvious that a column of any material will possess an absolute strength, the force necessary to break the column in tension. This force is proportional to the area of the cross-section (fig. 5, left). Now, Galileo is interested in obtaining the strength of a beam for which the absolute strength is known. Galileo chooses the simple case of a cantilever beam.



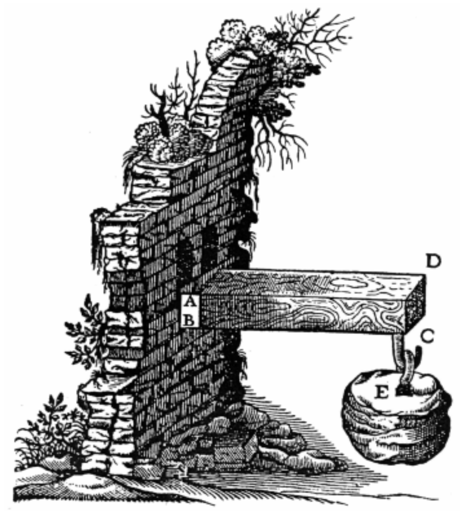


Fig. 5. *Left,* hypothetical trial to ascertain the absolute strength of a column. *Right*, the problem of the cantilever: How much weight could the cantilever sustain? [Galileo 1638]

He considers that the moment of resistance will be the result of multiplying the absolute strength by half the depth of the beam, as if a "hinge" (fulcrum) will form in the lower part of the critical section, where the beam is embedded into the wall. The analysis of Galileo is incorrect, as he forgets the necessary horizontal force to establish the equilibrium of horizontal forces: a member in bending must have both tension and compression areas. The correct solution was hinted at fifty years after by Mariotte in 1686; the first correct analysis was made by Parent in 1713; in 1773 Coulomb presented a complete description of simple bending theory. However, Galileo's error persisted in some handbooks until after 1800. The problem has been fully discussed in [Heyman 1998].

The square-cube law. However, Galileo was right about the form of the equation: for a given material and a certain cross-section, the bending strength is proportional to the product of its area by its depth. Galileo, then, applies himself to a comparison of the strength of beams of the same material and section but of different sizes. If the only load is the self-weight of the beam, Galileo realized that this load will grow with the third power of the linear dimensions, if the beam scales up maintaining its geometrical form. However, the strength will grow with the second power, the square, of the linear dimensions. As a consequence a structure becomes "weaker" as it grows in size, the reserve of strength diminishing linearly with size. If one wants to maintain the same strength, then, the cross-section must become thicker.

Galileo realized that this argument was a big discovery and immediately explained the consequences: "From what has already been demonstrated, you can plainly see the impossibility of increasing the size of structures to vast dimensions either in art or in nature". Therefore it will be impossible to build "ships, palaces, or temples of enormous size". Also, the size of an animal cannot be increased "for this increase in height can be accomplished only by employing a material which is harder and stronger than usual or by enlarging the size of the bones, thus changing their shape until the form and appearance of the animals suggest a monstrosity". Galileo then compares the deformation which will produce an increase of size of three times in the bone of an animal (fig. 6). The drawing explains the argument much better than the discussion in the text, and has been reproduced hundreds of times in texts about structures or biology.

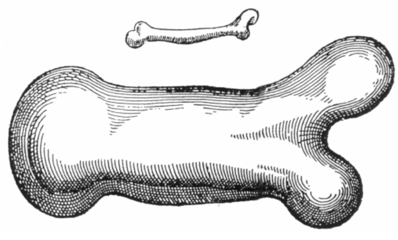


Fig. 6. The effect of an increase of size in the bones of an animal, if bone strength is considered to be constant

It is this reasoning which prompted Galileo's attack on proportional design at the beginning of his book. Galileo extends his conclusions about the strength of beams of different sizes to any structure, either natural or artificial. The argument is today called the "square-cube law" and it is still considered by many engineers as an irrefutable demonstration of the impossibility of proportional design. (For a discussion of the influence of scale in structural design see [Aroca 1999].

The shadow of Galileo. We said in the previous section that architects and engineers have always used proportional rules in masonry architecture design. If we accept Galileo's argument all of these rules are incorrect. Galileo's conclusion has been accepted by many historians of engineering and has conditioned in a negative way the appreciation of the traditional proportional design rules. For example, Parsons said :

> There were no means of testing materials to determine their resistance to strain and consequently, the designer could not estimate the strength of a member nor did he have a theory by which he would compute the amount of strain that a member would be called to bear. There was, therefore, a *vicious circle of ignorance* [Parsons 1939] (my italics).

And Benvenuto wrote:

> … il dimensionamiento in chiave geometrica restò sino a tempi recenti, il criterio più seguito dagli architetti: *il persistente pregiuduzio* che solo Galileo cominciò a smuovere, secondo il quale strutture geometricamente simili dovrebbero avere identiche proprietà statiche … aveva condotto numerosi tratattisti a definire in linguaggio geometrico la figura delle volte [Benvenuto 1981] (my italics).

Robert Mark, commenting on the design of Hagia Sophia, remarks on the same argument: "Geometry did play a major role in their conceptual design [of Hagia Sophia]; however, as no less an observer than Galileo also commented, geometry alone can never ensure structural success" [Mark 1990]. Some authors wonder themselves how it was possible, with such an erroneous approach, that the great buildings of the past were built; Harold Dorn wrote "… It is a tribute to their skill that with this assortment of anthropomorphic analogies, qualitative generalizations, traditional arithmetical proportions, rules-of-thumb and an intuitive (and incorrect) arch 'theory', Renaissance builders erected magisterial and lasting structures" [Dorn 1970].

A contradiction. Dorn hinted at the heart of the problem. If we consider as correct Galileo's argument we face a contradiction:

The great master builders of the past used proportional design rules, which are essentially incorrect.

Using these rules they built the masterpieces of architecture and engineering of the past.

It is not reasonable to believe that the masterpieces of historical architecture, which have survived for centuries or millennia, were designed following an incorrect approach. So, perhaps, the matter should be reconsidered.

In what follows we will make a short outline of the fundamental aspects of masonry structural design.

The design of masonry arches and vaulted structures

The matter may be best discussed with reference to the fundamental element of masonry architecture: the arch. We have seen that in a voussoir arch in equilibrium (see fig. 1 above), the stones transmit a thrust and that this thrust must be contained within the arch, to obtain a set of compressive stress equivalent to the thrust. The line obtained by joining the points of application of the thrust in every joint (the locus of these points) is the *line of thrust*. To understand the concept it is only necessary to have some familiarity with the parallelogram of forces. Traditionally two approaches have demonstrated its usefulness in arch analysis: the first is to consider the equilibrium of a semi-arch; the second is the analogy with the statics of hanging chains and cables.

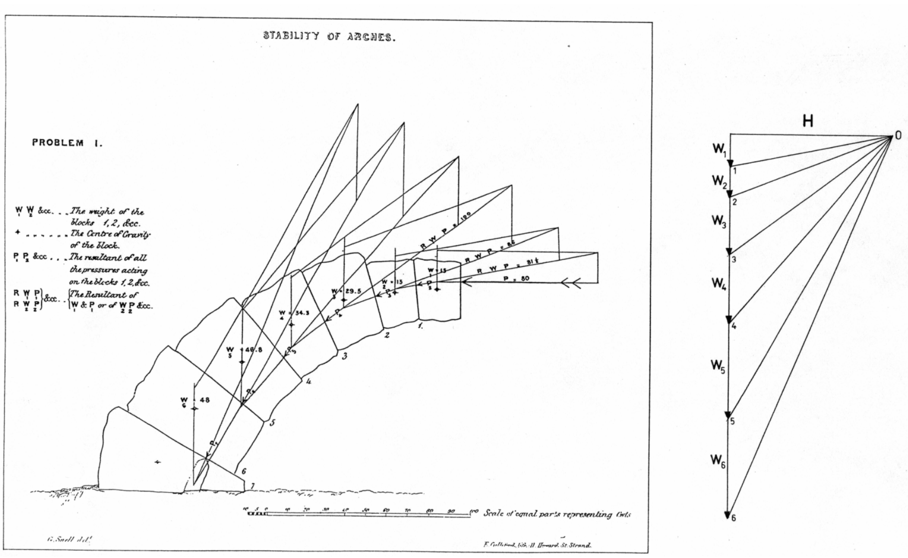


Fig. 7. *Left*, line of thrust in a semi-arch. The external horizontal thrust at the keystone is composed with the weights of the successive voussoirs defining a "path" of transmission of the forces [Snell 1846]. *Right*, the corresponding polygon of forces (added by the author)

Two half-arches: "A strength formed by two weaknesses". Consider the half-arch in fig. 7. It will clearly collapse, unless a force is applied in some point of the section at the keystone. If we apply an adequate horizontal thrust inside this section, then, as the figure shows, the thrust will be composed of the weight of every voussoir and the trajectory of the thrust will form the line of thrust drawn. In the original drawing the resolution of forces is made in the same drawing; however, as the horizontal thrust remains constant, all the lines may be gathered in one "force polygon" (added by the present author on the right of the figure). Then, using the terminology of graphical statics, the thrust line is an inverted funicular polygon, drawn tracing parallel lines to the force polygon. Note that in every joint the thrust is contained within the limits of the masonry. It is evident that by changing the value of the horizontal thrust we will obtain infinite lines of thrust within the arch. Also, we may change the point within the joint.

Now, we may imagine that we put another symmetrical semi-arch (a mirror reflection), on the other side. The horizontal thrust of the two semi-arches will equilibrate, no matter which line of thrust is considered: to use Leonardo's expression "an arch is a strength formed by two weaknesses". A semi-arch alone will collapse, but two "collapsing" semi-arches form a stable arch. It should be noted that the complete arch can be in equilibrium in infinite states of internal compression: in technical terms, the arch is statically redundant or "hyperstatic".

The cable analogy: "As hangs the flexible line . . .". Another way to understand the behaviour of masonry arches was proposed by Robert Hooke: "As hangs the flexible line, so but inverted will stand the rigid arch" [Hooke 1675] (fig. 8). The equilibrium of cables and arches is the same problem, and this was Hooke's genial analysis. Another English mathematician, David Gregory, completed Hooke's assertion: "None but the catenaria is the figure of a true legitimate arch, or fornix. And when an arch of any other figure is supported, it is because in its thickness some catenaria is included".

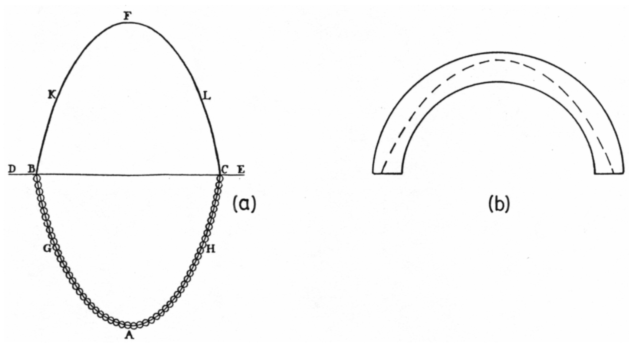


Fig. 8. (a) The arch as an inverted chain or cable. (b) The voussoirs of the arch may be imagined to be hanging from an imaginary chain, which represents the state of equilibrium. As the masonry must work in compression, the inverted "chain" must be contained within the arch [Heyman 1995]

The "material" masonry. The two previous analyses assume a material of certain properties, a "unilateral" material which can resist compression, but not tension. This condition "forced" the location of the thrusts within the masonry, to avoid tension. Traditional, unreinforced, masonry is just a pile of stones or bricks, disposed following some geometrical arrangement (the "bonding"), normally with mortar filling the joint or voids between the stones (sometimes there is no mortar). Roman concrete, where the volume of mortar is comparable to that occupied by the small stones, is also a type of masonry, and so is pisé (stiff earth or clay rammed until it becomes firm). In fact, any combination of stone, bricks, mortar or earth can lead to a successful kind of masonry. In fig. 9a, taken from a building handbook of ca. 1900, some types of masonry have been drawn. But in a common building we can find maybe a dozen different kinds of masonry (fig. 9b). Many times a structural element is a combination of several different masonries; this is typical of thick masonry walls with two external ashlar "shells" and a core made of rubble. Again, the number of combinations is almost unlimited.

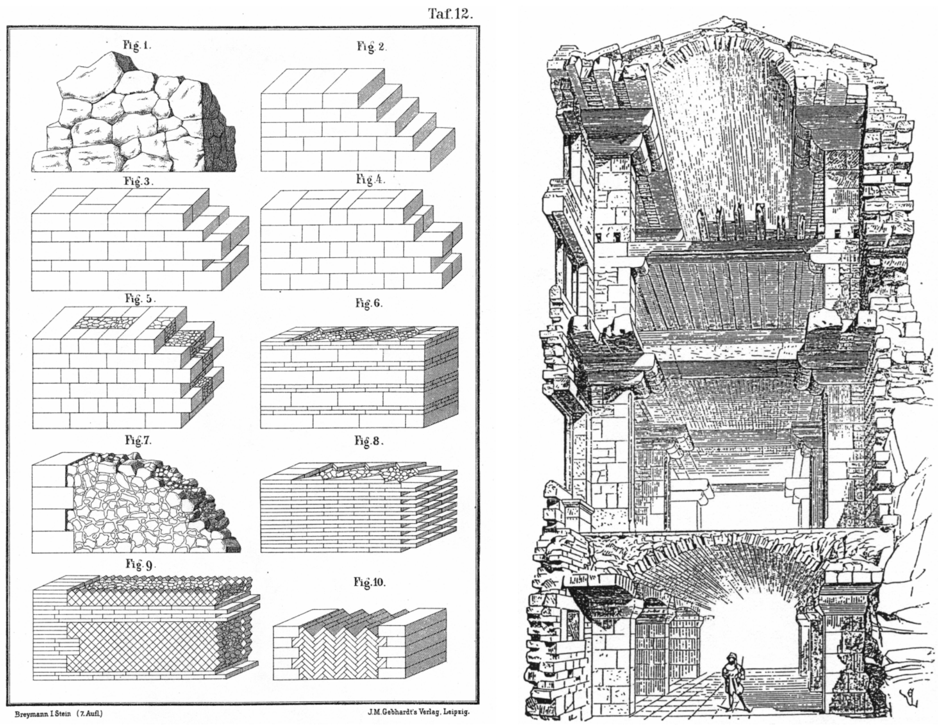


Fig. 9. (a) Different kinds of masonry [Warth 1903]. (b) Internal structure of a medieval building [Viollet-le-Duc 1854]

The question is: Where is the isotropic, homogeneous, elastic material of the classical elastic theory of structures? Nowhere. Masonry is essentially anisotropic, discontinuous, and heterogeneous; as for the elastic constants, one may ask, where?

In the external ashlar stone or in the rubble filling or in the mortar? It is easy to test a stone specimen to obtain, for example, the crushing strength. At the end of the nineteenth century thousands of tests were made on stone, brick and mortar. But when it came to ascertain the strength of the material "masonry" only vary vague indications, with enormous safety coefficients, were given (for stone tests see for example [Debo 1901]; for a critical discussion of the matter see [Huerta 2004]).

The essential characteristic of masonry is that it has good compressive strength and almost no tensile strength. Also, the stones maintain their position (no sliding occurs) due to the high friction coefficient (ca. 0.5). Heyman [1966, 1995] has systematized these observations into three Principles of Limit Analysis of Masonry: 1) Masonry has an infinite compressive strength; 2) Masonry has no tensile strength; 3) Sliding is impossible. These principles are reasonable and easy to check; they have been accepted, implicitly or explicitly, by all the masonry designers (architects and engineers) of the past centuries.

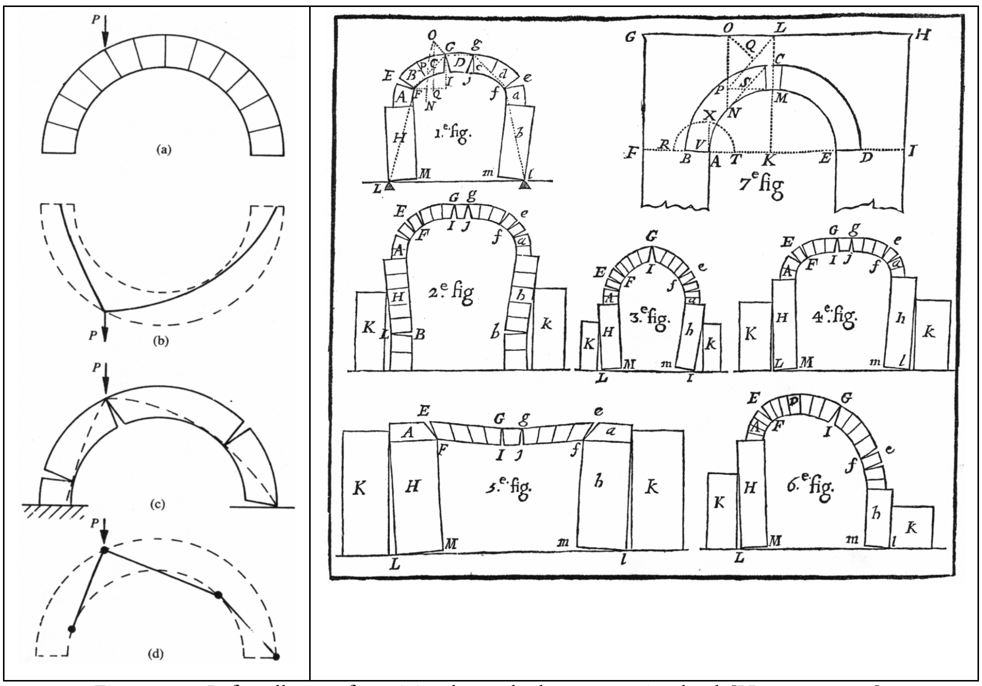


Figure 10. *Left*, collapse of semicircular arch due to a point load [Heyman 1995]. *Right*, first attempts to demonstrate the way of collapse of masonry arches [Danyzy 1732].

Collapse of masonry arches. If the material has these properties (and most types of masonry do), then the analysis of masonry structures may be included within the frame of Limit Analysis (or plastic theory) as Professor Heyman showed in 1966. The proof is complex, but the essence of the matter lies in that for a material of these properties, collapse will only occur when a sufficient number of hinges form which converts the structure into a cinematically admissible mechanism. The "hinges" are the points where the line of thrust touches the limit of the masonry.

Fig. 10, left, shows the process of collapse of an arch which supports an increasingly growing point load: the form of the "hanging chain" is modified until there is only one line of thrust inside the arch, touching alternatively the intrados or intrados; at the bottom the corresponding four-bar collapse mechanism is shown. The same applies to more complex structures. As a matter of interest, on the right side of the same figure appear the first essays made by the French scientist and engineer Danyzy in 1732, which demonstrate this mode of collapse.

The Safe Theorem. Now, within the frame of Limit Analysis, it can be demonstrated that if it is possible to draw a line of thrust within an arch, then this arch is safe, i. e., it will not collapse. This is a corollary of the Safe Theorem (or Lower Bound Theorem) of Limit Analysis. The theory of Limit Analysis, also called the Plastic Theory, was developed during the 1930s–1950s and it constitutes the fundamental contribution to the theory of structures in the twentieth century (in the same way as Elastic Theory was the main contribution in the nineteenth century). Professor Heyman has shown that the Fundamental Theorems, originally derived for steel frames, can be translated to masonry structures. His work has put the theory of masonry structures within the frame of the modern theory of structures; he has exposed with great clarity and intellectual rigour the consequences of this translation. The theory of plasticity itself is difficult, but, as sometimes occurs, the consequences are quite easy to understand. For example, the condition that the line of thrust must be contained within the arch leads to purely geometrical statements.

The limit arch. An arch of sufficient thickness will contain infinite lines of thrust, for example the arch in fig. 11a. If we reduce the thickness of the arch, the form of the line of thrust will suffer no change, but it is evident that for a certain thickness only one line will be contained within the arch: this arch is the *limit arch* and its thickness is the *limit thickness* (fig. 11b).The limit thickness can be expressed conveniently as a certain fraction of the span. For a semicircular arch the limit thickness is nearly 1/18 of the span. That means that a masonry arch thinner than this proportion cannot be built; the arch will become a mechanism, collapsing (fig. 11c). Thus, the limit arch forms the point of departure for the design of a safe arch: we will obtain geometrical safety by "thickening" the limit arch.

There are two approaches for the design of a safe arch: the "strength" approach and the "stability" approach. In both approaches the limit arch is the point of departure. If we want to follow a condition of strength, the thickness should be increased until the stresses reach a certain "admissible" value obtained dividing the crushing stress by a certain coefficient. If we are concerned with a possible failure by lack of stability (the formation of a collapse mechanism), then we increase the thickness multiplying it by a certain *geometrical factor of safety* (this concept has been introduced by [Heyman 1969]). In the case of arches, a typical value is 2 or 3; so a safe arch will have double the thickness of the limit arch.

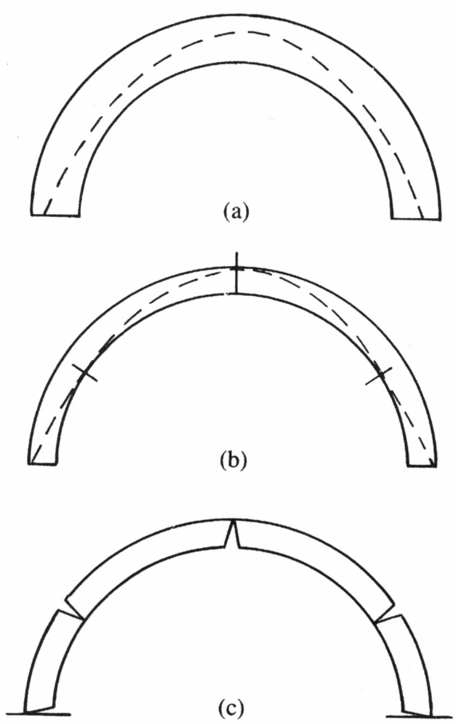


Fig. 11. (a) An arch of sufficient thickness contains comfortably infinite lines of thrust. One has been drawn. (b) Diminishing the thickness, we arrive at the limit thickness, where only one line can be drawn within the arch. (c) At t he points where the line touches the limit of the masonry a "hinge" forms. The arch is in mathematical equilibrium [Heyman 1995]

Masonry arch design: Strength versus Stability. The problem is which condition governs the design. Maybe the best thing is to take an example. Consider an arch of stone of 18m span. The limit arch will have a thickness of very nearly 1m.

Strength: if the arch is going to be made of medium sandstone (20 kN/m^3) with an admissible working stress of, say, 4 N/mm² (1/5 of a crushing strength of 20 N/mm²). Then, it is easy to calculate (considering a uniform stress distribution) that the required increase of thickness will be of 80mm or 0.5% of the span. (In fig. 12b the increase of thickness has been exaggerated: in fact the increase will be within the thickness of the lines of intrados and extrados in fig. 12a.) The arch is so near the limit thickness that it is on the verge of collapse; in fact, by inspection, the overall form of the arch has not changed.

Stability: the usual geometrical factor of 2 or 3 (the last value represented in fig. 12c) will impose a substantial change in the form of the arch, which will be easily recognised by inspection. Any master mason will know, just seeing it, that the arch is not only safe, but that it has a surplus (the usual geometrical factor of safety for arches is 2).

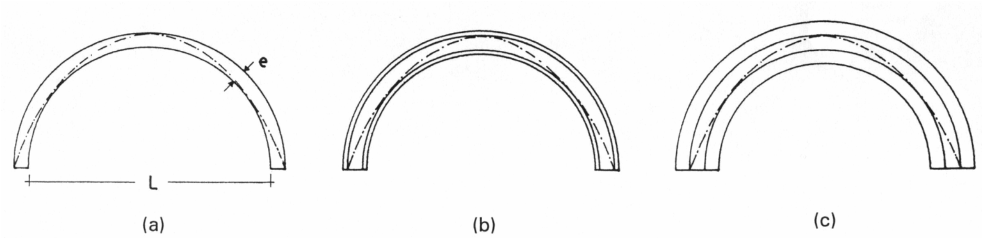


Fig. 12. Design of a masonry arch. (a) The limit arch; (b) Design by "strength"; the increase of thickness should guarantee admissible stresses; (c) Design by "stability"; the increase of thickness should afford a geometrical safety to the arch

The strength criterion is unsafe: the stresses will be low but the arch will be dangerously near of the collapse situation. Of course, the calculations have been made for an arch of 18m. For greater arches the stresses will grow linearly, and the increase of thickness will be correspondingly greater.

Limit spans for masonry. For a certain span, the thickness by strength will coincide with the thickness by stability and this point will mark the limit of the span of the arch. For a geometrical factor of safety of 2 (and considering a rectangular block of stresses at the base) this maximum span will be:

$$s_{\max} = \frac{2}{\pi} \frac{\sigma_{adm}}{\gamma}$$

where σ_{adm} is the maximum admissible working stress for the material and γ is the specific weight of the masonry. For the above stated data, $s_{\max}$ = 128m and the absolute maximum span for the crushing strength will be 5 times this span, or 640m. With stones of a better quality, correspondingly greater spans may be built. The dimensions are well above the usual dimensions of bridges. The largest span of an stone arch bridge is that of Fong-Huan, in China, built in 1972, with 120m [Fernández Troyano 1999]. In concrete (with no longitudinal reinforcement) it is the bridge of Caille in Cruseilles (1928) with 139.8m. The quantity σ_{adm}/γ, which is a length, represents the limit height of a column of uniform section built with this material; this quantity was used in the nineteenth century as a measure of the strength of the materials and, also, as an indication of the maximum sizes which can be attained (fig. 13).

INDICATION DES MATÉRIAUX.	Poids du décimètre cube.	Charge d'écrasement par centimètre carré.	Hauteur représentative de la charge d'écrasement (1).	OBSERVATIONS.
Pierres volcaniques.	kilogr.	kilogr.	mètres.	
Basalte de Suède.	3,06	1912	6248	Rondelet.
Basalte d'Auvergne.	2,88	2078	7215	*Id.*
Lave du Vésuve, dite *Piperno*. . .	2,60	563	2165	*Id.*
Lave grise des environs de Rome.	1,97	228	1157	*Id.*
Tuf de Rome.	1,22	58	478	*Id.*
Granits.				
Granit d'Aberdeen bleu.	2,62	767	2927	G. Rennie.
Granit vert des Vosges..	2,85	620	2175	Rondelet.
Granit gris de Bretagne.	2,74	654	2383	*Id.*
Granit de Normandie, Gatmos. . .	2,66	702	2639	*Id.*
Granit gris des Vosges.	2,64	423	1603	*Id.*
Grès.				
Grès très-dur.	2,52	813	3226	*Id.*
Grès blanc.	2,48	923	3713	*Id.*
Grès bigarré des Vosges.	2,17	400	1843	Conservatoire des arts et métiers.
Pierres calcaires.				
Marbre noir de Flandre..	2,72	789	2901	Rondelet.
Marbre blanc veiné.	2,70	298	1104	*Id.*
Marbre rouge du Devonshire.. . .	2,70	522	1933	Rennie.
Calcaire de Portland..	2,42	262	1083	*Id.*
Pierre de Caserte, près Naples. . .	2,72	595	2191	Rondelet.
Pierre noire de St-Fortunat (Lyon)	2,65	627	2366	*Id.*
Liais de Bagneux, près Paris. . .	2,44	445	1824	*Id.*
Travertino de Rome.	2,36	298	1262	*Id.*
Roche de Châtillon, près Paris. . .	2,29	174	760	*Id.*
Roche douce de Châtillon.	2,08	134	644	*Id.*
Roche d'Arcueil, près Paris. . . .	2,30	253	1100	*Id.*
Pierre de Saillancourt, 1re qualité.	2,41	141	585	*Id.*
Briques.				
Brique dure très-cuite.	1,55	150	962	
Brique rouge.	2,17	57	262	
Brique rouge pâle.	2,08	39	187	
Mortiers.				
Mortier de chaux et de sable de rivière.	1,63	31	»	Rondelet.
Mortier de ciment de tuileau. . .	1,46	48	»	*Id.*
Mortier de pouzzolanes de Naples et de Rome mêlées.	1,46	37	»	*Id.*
Mortier avec chaux éminemment hydraulique.	»	144	»	Vicat.

(1) Cette colonne indique la hauteur du prisme droit de la matière considérée dont le poids serait suffisant pour écraser sa propre base (§ 25).

Fig. 13. Table of the strength of stones and bricks. On the second column from the right, the limit height in meters has been calculated [Collignon 1885]

Engineers of the past were well aware of the possibility of building great spans in masonry. The bridge over the Adda in Trezzo, built in 1370–77 (demolished in 1416 for military reasons) had a span of 72 m (this span was surpassed only after 1900). But perhaps the best example of the confidence that enormous spans can be made is found in a design of Leonardo da Vinci, ca. 1500, for a bridge of one arch over the Golden Horn in Istanbul (fig. 14, left). Leonardo's bridge would have had a span of 240 m and in the manuscripts he shows concern only for the problems of centering. Stüssi [1953] undertook an exhaustive analysis of Leonardo's design and concluded that it would have been feasible. Stüssi obtained a maximum stress of 10 N/mm^2 for a material with a specific weight of 28 kN/m^3 (fig. 14, right).

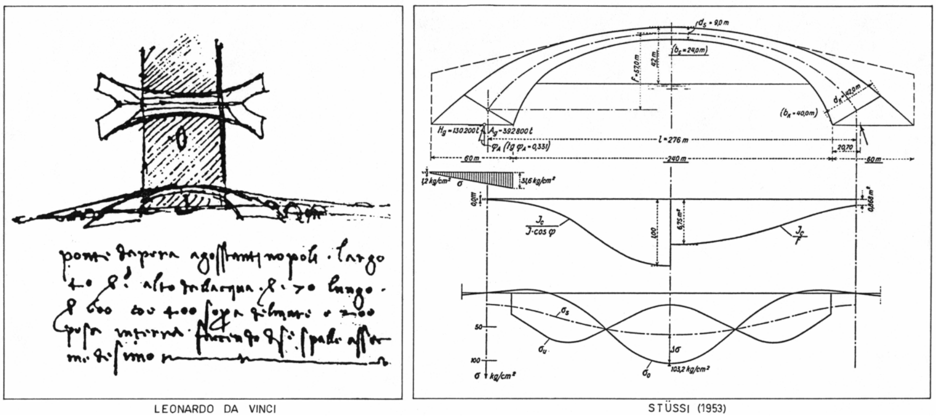


Fig. 14. *Left*, Leonardo de Vinci's design for an arch bridge with a span of 240 m, over the Golden Horn in Istanbul (ca. 1500). *Right*, Stüssi's analysis

Stresses in masonry buildings. In buildings the same thing occurs. Even in the greatest structures built the calculated mean stresses in the most loaded parts (in general the columns) are still quite moderate, as can be seen in Table 1 below. For example, in the main piers of St. Peters, which supports a dome and drum with a total weight of 400,000 kN, the mean stress is 1.7 N/mm^2. A similar structure three times bigger could have been built, but would it have had any meaning?

In conclusion, it is a fact that for historical masonry structures, the stresses are an order or two orders of magnitude below the crushing strengths of the masonry and, therefore, the problem of masonry design is not governed by strength but by stability.

Stability governs the design, which means that the objective is to design safe formsl. Considerable savings of material may be obtained by choosing adequate "geometries". Economy was, of course, the second main structural concern of old master builders. Until recent times, the search for economical structures and economical building procedures has been a constant. Choisy showed his surprise when he discovered in 1873 that precisely this striving for economy was the key to a deeper understanding of the Imperial Roman building processes.

BUILDINGS	*Mean stress* *N*/mm²
Columns, church of Toussaint d'Angers	4.4
Main pillars, French Pantheon (St. Genevieve), Paris	2.9
Main pillars, Hagia Sophia	2.2
Main pillars, cathedral of Palma de Mallorca	2.2
Main pillars, St. Paul, London	1.9
Main pillars, St. Peters. Rome	1.7
Main pillars, Church des Invalides , Paris	1.4
Main pillars in the cathedral of Beauvais	1.3
Base of the tambour of the Roman Pantheon	0.6
BRIDGES	
Bridge of Morbegno (s = 70 m)	7.0
Bridge of Plauen (s = 90 m)	6.9
Bridge of Villeneuve (s = 96 m)	5.7
Viaduct of Salcano, Göritz (s = 85 m)	5.1
Bridge over the Rocky River (s = 85 m)	4.4
Bridge of Luxemburg (s = 85 m)	4.8

Table 1. Mean stresses in some of the biggest masonry structures. In almost every case the mean stress is at least an order of magnitude below the crushing strength of the corresponding masonry [Huerta 2004]

Geometry and structural economy: two case studies of arch design. As an example of the importance of design for considerable savings in material without a diminution of the safety, a simple case will be investigated: the design of a simple barrel vault on rectangular buttresses. The first thing is to design the vault with an adequate geometrical safety. For a semicircular arch the limit thickness is nearly t = 1/18. But for segmental arches with an opening angle of less than 180° the limit thickness diminishes very rapidly (fig. 15, left).

The design strategy to use "thin" vaults safely is to reduce the height of the arch, filling the haunches of the vault with good masonry up to a certain height. In fig. 15, above, two designs have been drawn of vaults with the same geometrical safety factor of 3 (thickness 3 times the limit): a good filling of the haunches up to half the height of the vault permits a reduction of the division of the thickness of the vault by 4. The thrust of the vault is, therefore, also divided by nearly the same factor and the buttresses (for a height equal to the span) may be reduced from nearly 1/3 of the span to nearly 1/4 of the span. The total amount of masonry (vault plus buttresses) is reduced by 40%. This remarkable economy is the result of just putting some rubble masonry between the haunches and the wall.

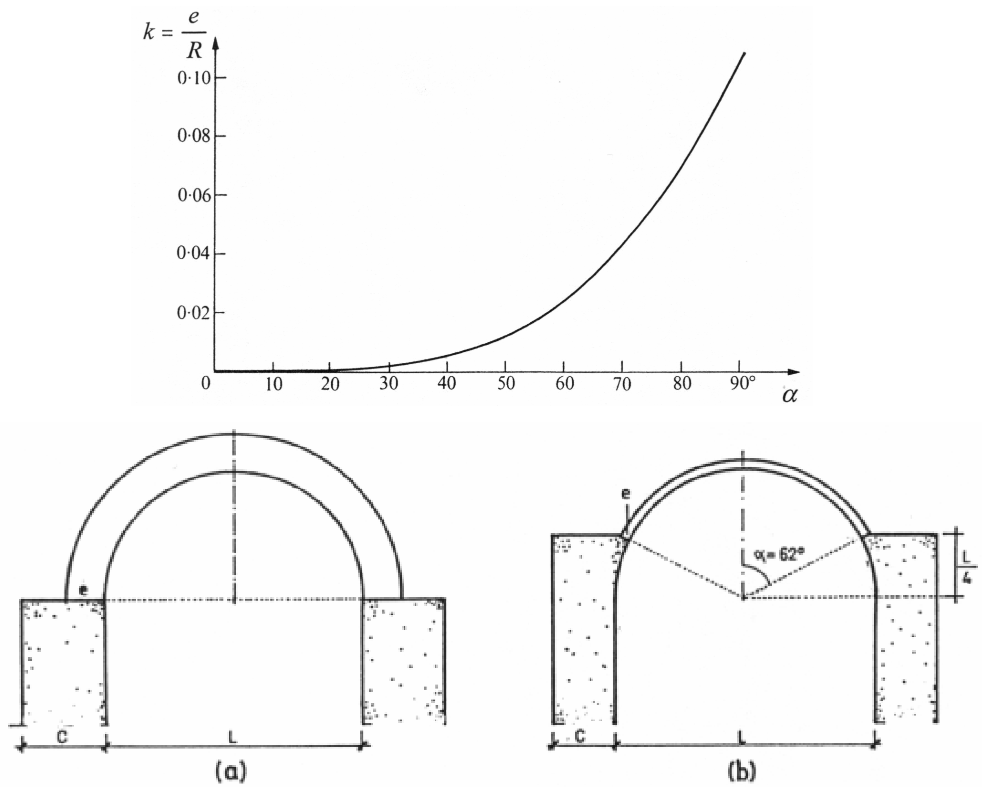


Figure 15. *Above*, limit thickness of circular segmental angles, depending on the opening semi-angle [Heyman 1995]. *Below*, effect of the reduction of the opening angle by filling the haunches with good masonry [Huerta 2004].

Another example of the same kind of geometrical design is that of pointed arches. A pointed arch has, in general, a smaller limit thickness than the corresponding semicircular arch of the same span. However, a simple filling of the haunches will not reduce the limit thickness with the same rapidity as in a semicircular arch, due to the solution of continuity at the top. In fact, to achieve good results a pointed load is needed at the tip of the arch. This load will "break" the smoothness of the line of thrust, better adapting it to the form of the arch. In the drawing by Hatzel (fig. 16, left) on the left side the line of thrust has been drawn; it is evident that it will be impossible to introduce it within the arch whose thickness is less than the limit thickness. However, on the right side a weight has been added on top; now the line of thrust coincides almost perfectly with the middle line and goes out in the lower half, where some filling should be put on the haunches to allow the thrust pass to the buttress system.

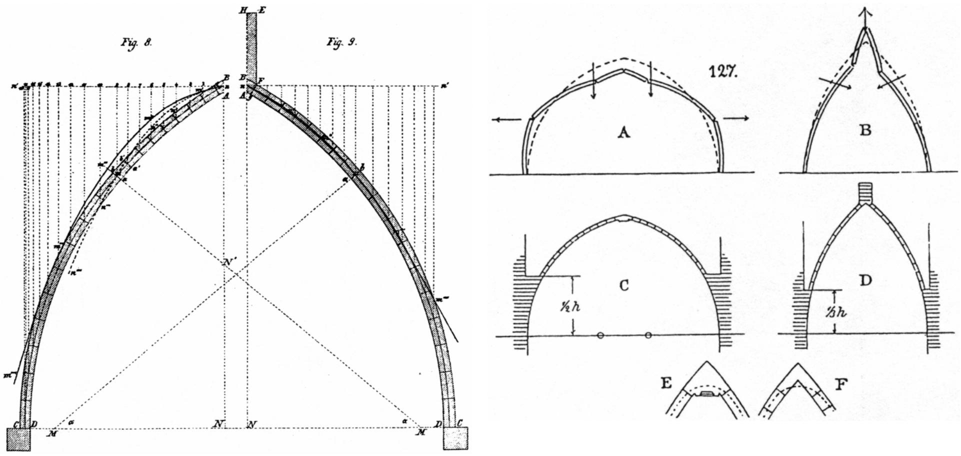


Figure 16. *Left*, equilibration of a pointed arch by adding a weight on its top [Hatzel 1849]. *Right*, collapse of pointed arches of insufficient thickness and their equilibration by loading them adequately [Ungewitter 1890]

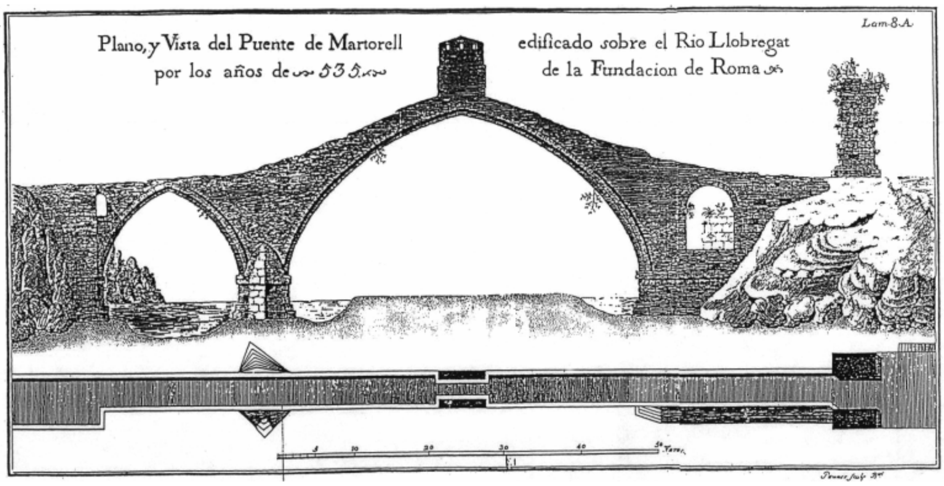


Fig. 17. Puente del Diablo (devil's bridge) in Martorell; thirteenth century, on Roman springings. The tower at the top of the bridge helps to stabilise the thin ring of the arch (after [Sánchez Taramas in Muller 1769])

This was well known by the Gothic master builders and the heavy, sometimes richly sculpted, keystones of pointed arches and cross vaults serve a structural as well as a decorative function. Tosca, commenting on the design of pointed medieval arches, said: "These arches correspond to the Gothic order, and though being beautiful, they are weak near the haunches … particularly if they have no load on the keystone" [Tosca 1707]. Mohrmann was also well aware of this when explaining the statics of medieval arches in his 1890 edition of Ungewitter's manual and his drawings explain clearly Tosca's assertion (fig. 16, right).

Bridge builders were also aware of this. The medieval Puente del Diablo (devil's bridge) in Martorell shows a little tower on the keystone (fig. 17). The construction serves to control the passage through the bridge but it also plays a fundamental role in stabilising the thin ring of the arch. If, for example, during some work of restoration the tower were to be removed to be re-built afterwards, this may prove to be a dangerous operation.

The "ideal" arch. Now we may make a short digression about the "ideal" form of the arch. Is it the catenary, as was maintained by Hooke and Gregory, and later by many others up until the present day? In a catenarian arch all the voussoirs are different as the radius of curvature changes from point to point. It is difficult to build the centering and if the arch is made of stones, every stone will need a different template. The Gothic approach is cleverer: you choose a geometrical form made of circular arcs and then you load this arch to make it "catenarian". The rubble filling and the stone at the top have almost no cost. A "catenarian" architecture like that of Gaudí is, in fact, quite expensive in comparison with Gothic architecture. We will leave the matter here, because to pursue it we must enter in the realm of architectural design which, of course, involves many other aspects besides that of structural efficiency (for a discussion of Gaudí's structural design see [Huerta 2003]).

A wonder of equilibrium: the cathedral of Palma de Mallorca. Up to now we have discussed just simple examples of the kind of geometrical design which is characteristic of masonry architecture. Of course, in a building of some scale many different structural problems are present and the master builder must take into account all of them and, eventually, produce an integrated design. The degree of subtlety which can be found in some projects is amazing. A good example is that of the cathedral of Palma de Mallorca (fig. 18).

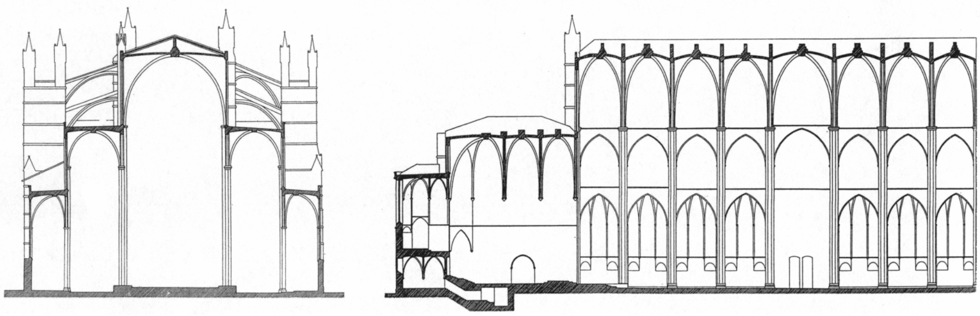


Fig. 18. Cross and longitudinal section of the cathedral of Palma de Mallorca [Domenge 1999]

This is one of the biggest Gothic cathedrals: the main nave has a span of 20m and a height of 42m. The nave columns are extraordinary slender and must support at the top the thrust of the lateral aisles. How is it possible for such a slender column to function as a buttress?

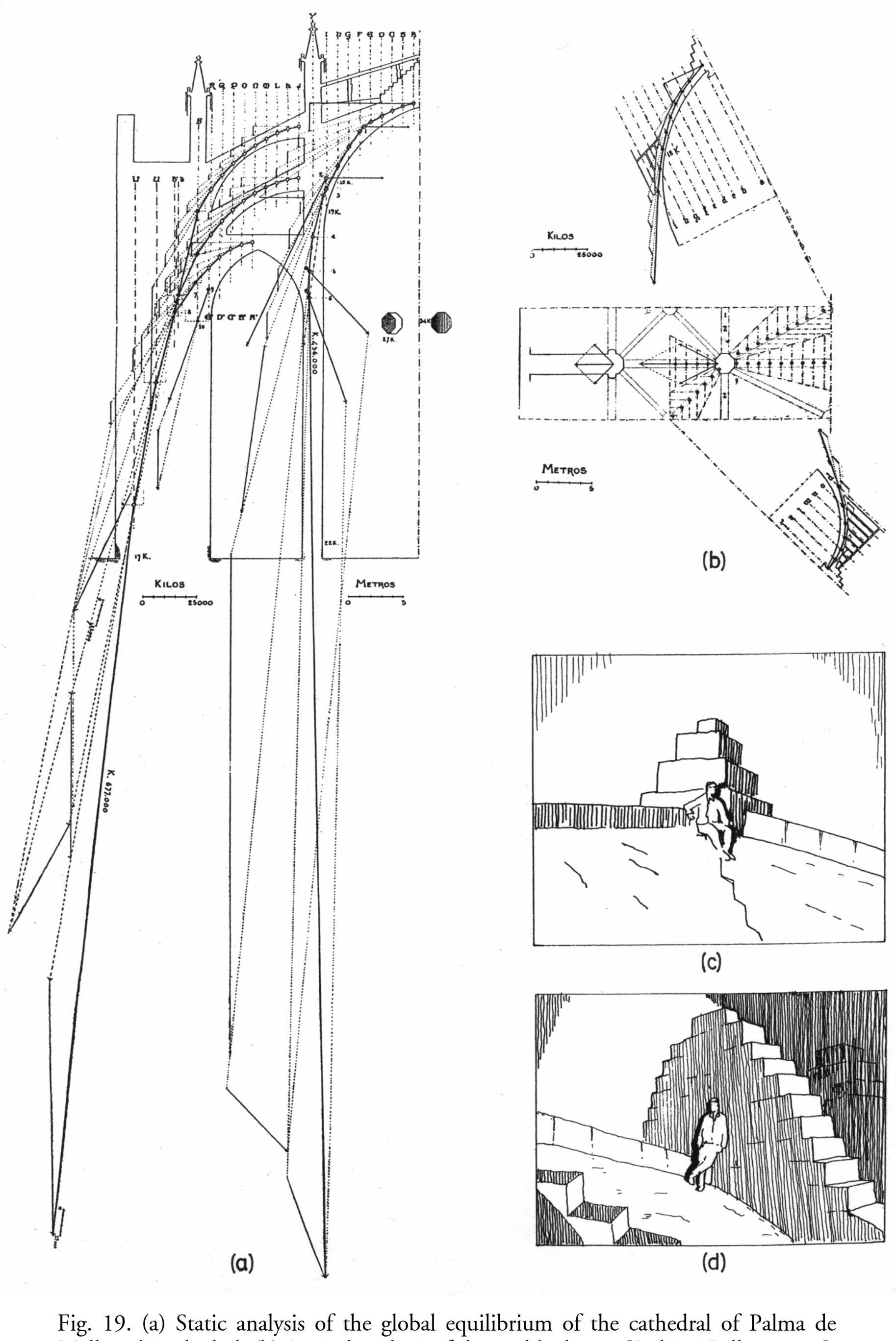


Fig. 19. (a) Static analysis of the global equilibrium of the cathedral of Palma de Mallorca's cathedral. (b) Statical analysis of the vault's thrusts [Rubió i Bellver 1912]; (c) and (d) Loads on top of the crossing and transverse arches (redrawn by the author after [Rubió i Bellver 1912])

The question cannot be answered without climbing to the extrados of the high vaults. There one may see pyramids of stone on top of the crossing arches, heavy transverse walls on the transverse arches, extraordinarily thick ribs, etc. The unknown master used a typical Gothic device: buttressing by loading. The weight on top of the columns is increased extraordinarily so that the thrust of the lateral aisle only deviates slightly the vertical direction of the loads. Of course, these extraordinary loads produce a lateral thrust and the external buttresses which receive it are the biggest of all Gothic architecture, with a depth (8m) approaching one half of the span. In 1912 Rubió i Bellver, a disciple of Gaudí, published a static analysis of the cathedral and this analysis confirms the qualitative comments made before. In Rubio's drawing of the trajectory of the thrusts, with the different forces drawn to scale, it is easy to see the equilibrating effect of the added loads (fig. 19).

We have considered the problem of masonry arch design in some detail in order to show the importance of the geometry on the arch's safety. The example of Palma de Mallorca illustrates how this kind of geometrical design is at the heart of masonry design. This depends on the Safe Theorem of Limit Analysis which, as Professor Heyman has said, is "the rock on which the whole theory of structural design is now seen to be based" [Heyman 1999]. The main corollary of this theorem leads to the "approach of equilibrium" and, for a masonry structure, the problem of obtaining a compressive state of equilibrium is a geometrical problem. Citing again Heyman: "The key to the understanding of masonry is to be found in a correct understanding of geometry" [1995].

Conclusion: The "error" of Galileo and "Navier's straitjacket"

Galileo was the first to provide a theory which permitted the strength of a certain type of structural element – the simple beam – to be checked. He was the founder of the theory of structures. He was right in deducing that the strength of a certain section of a beam is proportional to the strength of the material and to the area and depth of its cross-section. Also correct is his observation that, given any structure which supports its own weight, if we multiply its "size" by a certain factor, maintaining the geometrical form of the structure, the loads grow with the cube of the factor, but the sections of the structural members grow with the square, and as their strength are proportional to the areas, either the structure becomes "weaker" or the members must be thickened. In modern terms: stresses grow linearly with the dimensions, that is, they are directly proportional to the scale factor. All this is completely correct and it is an extraordinary feat of genius that Galileo, working alone in his old age, not only founded the New Science of the Strength of Materials, but that also drew design conclusions: the square-cube law.

Galileo realized that his discovery contradicted a traditional design approach, or rule: that of proportional design. He was well aware that it was an important discovery, which would affect the design of many types of structures: machines, ships, beam and framed structures. The theory also permitted an explanation of some biological facts: why the bones of small animals are proportionally slender and, also, why small animals are proportionally stronger. He expressed his discovery so convincingly that his argument has reached the rank of "Law" in the books on structural design: "the square-cube law".

The law refers only to problems of the design of structures supporting mainly their own weight when the governing criterion is strength. This is, indeed, one of the three fundamental structural criteria: strength, deformation and stability. A structure should resist its loads without the breaking of any of its members. It also should not present unduly large deformations. Finally, the structural elements and the structure as a whole should not be unstable.

In modern structures strength is usually the governing criterion. But, as we have seen, in historical constructions strength plays no role and it is stability which is relevant. Galileo was perhaps too quick to generalize his discovery to any structure, including specifically "palaces or temples". He was wrong in applying his "strength" argument to masonry structures. But it is understandable that a scientist should look for universal laws. It is also understandable that he should be carried away by enthusiasm for a great discovery.

But after more than three centuries it is remarkable that Galileo's argument continues to be applied uncritically to structures where it is evident, in the etymological sense, that it does not apply, by simple comparison of structures of different sizes. In fact, any reader of books on the history of architecture would have problems in ascertaining the actual size of a building from a plan without scale. If we compare the form of the domes of San Biaggio in Montepulciano, St. Peters in Rome and Santa Maria del Fiore, drawing the three at the same size, we may see that the overall form and proportions are very similar (fig. 20).

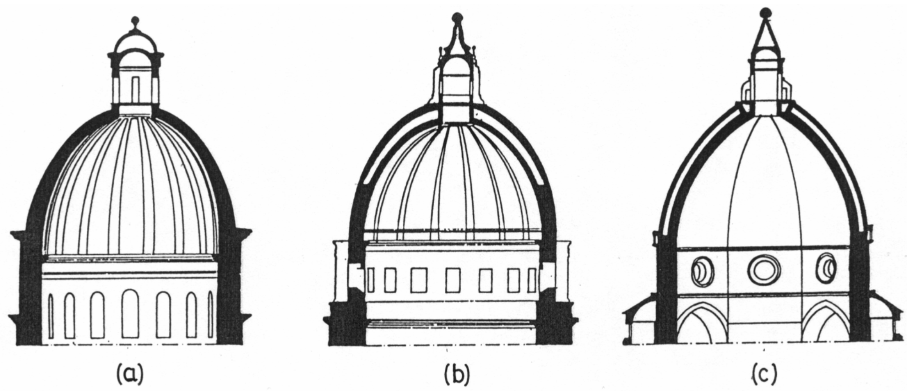


Fig. 20. Comparison of the form of three Renaissance domes: (a) San Biaggio in Montepulciano (14m); (b) St. Peters in Rome (42m); (c) Santa Maria del Fiore (42m). Note that, although the first dome is three times as small than the other two the form is very similar

However, the dome of San Biaggio has a span of 14m, 1/3 of the 42m of both St. Peters and Santa Maria del Fiore. Byzantine architecture is full of domes similar to that of Hagia Sophia and the geometry of cupola of the Pantheon has been reproduced hundreds of times.

Now the question is why an argument which doesn't coincide with the facts (the design of masonry architecture) has been used for three centuries and continues to be used. The first reason has been already mentioned: the argument *is true.* For any structure subject to its own weight, including masonry structures, internal stresses grow linearly with size in similar structures. What is not true is that stresses (strength) determine the design of masonry structures.

The emphasis on strength and the opposition of the "actual" internal stresses in structures as the main objective of structural theory comes from the very development of the science of structures. Navier in his *Leçons* of 1826 stated explicitly the aims of the structural theory: the resolution of the three structural equations, those of equilibrium, of the elastic material and of compatibility, will give a unique solution for the internal forces within the structure. Then internal stresses will be calculated from them and, finally, these stresses will be compared with values of the strength of the material obtained in experimental tests. The focus is on obtaining the stresses and the complicated mathematical apparatus of the theory of elasticity and of the resolution of the system of equations precluded for almost one hundred years any criticism. Professor Heyman has called this frame of reference "Navier's straitjacket" [1999]. Nowadays the Method of Finite Elements, the numerical resolution of the system of the three structural equations dividing the structure in "finite elements", points in the same direction as the "old" classical elastic theory. Sometimes, of course, this has negative consequences in the field of structural intervention on historical buildings. As we have seen, many times the only way to assure the safety of a building is to "overload" some of its parts, as it occurred with the main columns in the cathedral of Palma de Mallorca. A reduction of weight, which "theoretically" always leads to a reduction of stresses, may lead to serious damage and, eventually, to the collapse of the structure.

In summary, any engineer or architect with some formation in structural theory feels more comfortable within the frame of the strength approach of Galileo and the classical theory of structures. It requires an effort, and some study, to overcome our own prejudices and to accept that, for example, the medieval master masons, knowing nothing of mathematics, elastic theory and strength of materials, had a deeper understanding of masonry architecture than we engineers and architects of the twenty-first century do. However the masonry buildings of the past stand today as a proof of their knowledge and our ignorance.

References

AROCA HERNÁNDEZ-ROS, Ricardo. 1999. *¿Qué es estructura?* Madrid: Cuadernos del Instituto Juan de Herrera.

BENVENUTO, Edoardo. 1981. *La Scienza delle Construzioni e il suo sviluppo storico.* Florence: Sansoni.

BESENVAL, Roland. 1984. *Technologie de la voûte dans l'Orient Ancien.* 2 vols.Paris: Editions Recherche sur les Civilisations.

COENEN, Ulrich. 1990. *Die spätgotischen Werkmeisterbücher in Deutschland. Untersuchung und Edition der Lehrschriften für Entwurf und Ausführung von Sakralbauten. (Beiträge zur Kunstwissenschaft, Bd. 25).* München: Scaneg.

COLLIGNON, Édouard. 1885. *Cours de mécanique appliquée aux constructions. Première partie: Résistance des matériaux.* 3rd. ed. Paris: Vve. Ch. Dunod.

DEBO, Ludwig. 1991. *Lehrbuch der Mauerwerks-Konstruktionen.* Hannover: Gebruder Jänecke.

DOMENGE I MESQUIDA, Joan. 1999. La catedral de Mallorca: Reflexiones sobre la concepción y cronología de sus naves. In: *Gotische Architektur in Spanien*, ed. C. Freigang, 159–187, 395–398. Frankfurt a. M./Madrid: Vervuert/Iberoamericana.

DORN, Harold I. The Art of Building and the Science of Mechanics. A Study of the Union of Theory and Practice in the Early History of Structural Analysis in England. Ph.D. thesis, Princeton University, 1970.

DURM, Josef. 1885. *Die Baukunst der Etrusker und Römer. (Handbuch der Architektur. Zweiter Teil. 2.Band).* Darmstadt: J. Ph. Diehl.

EL-NAGGAR, Salah. 1999. *Les voûtes dans l'architecture de l'Égypte ancienne.* 2 vols. Cairo: Institut Français d'Archéologie Orientale.

FERNÁNDEZ TROYANO, Leonardo. 1999. *Tierra sobre el agua. Visión histórica universal de los puentes.* Madrid: Colegio de Ingenieros de Caminos Canales y Puertos.

GALILEI, Galileo. 1638. *Discorsi e Dimostrazioni Matematiche intorno à due nuove sicenze Attenenti alla Mecanica & i movimenti Locali.* Leiden: Elsevier. (English translation by Henry Crew and Alfonso de Salvio, New York: Dover, 1954.)

GREGORY, D. 1697. Catenaria. *Philosophical Transactions of the Royal Society* **19**, 231: 637–652.

HATZEL, E. 1849. Über die Technik in spezieller Beziehung auf die Architektur und die Gestaltung der Formen. *Allgemeine Bauzeitung,* 132–169.

HECHT, Konrad. 1979. *Maß und Zahl in der gotischen Baukunst.* Hildesheim: Georg Olms Verlag.

HEYMAN, Jacques. 1966. The Stone Skeleton. *International Journal of Solids and Structures* **2**: 249–79.

———. 1998. *Structural analysis: a historical approach.* Cambridge: Cambridge University Press.

———. 1995. *The Stone Skeleton. Structural Engineering of Masonry Architecture.* Cambridge: Cambridge University Press.

———. 1999. *The Science of Structural Engineering.* London: Imperial College Press.

———. 1999. Navier´s straitjacket. *Architectural Science Review* **42**: pp. 91-95.

HOOKE, Robert. 1675. *A description of helioscopes, and some other instruments.* London.

HUERTA, Santiago. 2003. El cálculo de estructuras en la obra de Gaudí. *Ingeniería Civil* **130**: 121–33.

———. 2004. *Arcos, bóvedas y cúpulas. Geometría y equilibrio en el cálculo tradicional de estructuras de fábrica.* Madrid: Instituto Juan de Herrera.

———. 2006. Geometry and equilibrium: The gothic theory of structural design. *Structural Engineer* **84**: 23–28.

KOEPF, Hans. 1969. *Die gotischen Planrisse der Wiener Sammlungen.* Vienna: Hermann Böhlaus Nachfolger.

MARK, Robert. 1990. *Light, wind, and structure.* Cambridge, MA: MIT Press.

MULLER, Juan. 1769. *Tratado de Fortificación ó Arte de construir los Edificios Militares, y Civiles … traducido y aumentado con notas de D. Miguel Sánchez Taramas.* Barcelona: Thomas Piferrer.

NAVIER, C.L.M.H. 1826. *Résumé des Leçons donées à L'École des Ponts et Chaussées, sur l'application de la mécanique à l'établissement des constructions et des machines*. Paris. (2nd. ed. 1833; 3rd ed. with notes and appendices by B. de Saint-Venant 1864.)

PARSONS, W.B. 1976. *Engineers and Engineering in the Renaissance*. Cambridge, MA: MIT Press. (1st ed. 1939.)

RUBIÓ I BELLVER, Joan. 1912. Conferencia acerca de los conceptos orgánicos, mecánicos y constructivos de la Catedral de Mallorca. *Anuario de la Asociación de Arquitectos de Cataluña* (1912): 87–140.

SNELL, George. 1846. On the Stability of Arches, with practical methods for determining, according to the pressures to which they will be subjected, the best form of section, or variable depth of voussoir, for any given extrados or intrados. *Minutes and Proceedings of the Institution of Civil Engineers* 5: 439–476, Plates 27–40.

TOSCA, Tomás Vicente. 1707–15. *Compendio mathemático en que se contienen todas las materias más principales de las ciencias que tratan de la cantidad …* Valencia: Antonio Bordazar.

UNGEWITTER, Georg Gottlieb. 1890. *Lehrbuch der gotischen Konstruktionen. III Auflage neu bearbaitet von K. Mohrmann.* 2 vols. Leipzig: T.O. Weigel Nachfolger.

VIOLLET-LE-DUC, Eugene. 1854–1868. *Dictionnaire raisonnée de l'Architecture Française du XI au XVI siécle*. 10 vols. Paris: A.Morel.

WARTH, Otto. 1903. *Die Konstruktionen in Stein*. Leipzig: J. M. Gebhardt's Verlag.

About the author

Santiago Huerta became an architect in 1981 following study at the School of Architecture of the Polytechnic University of Madrid. He was in professional practice from 1982 to 1989. In 1989 he became Assistant Professor in the School of Architecture of Madrid. He earned a Ph.D. in 1990 with a dissertation entitled "Structural design of arches and vaults in Spain; 1500-1800". Since 1992 he has been Professor of Structural Design at the School of Architecture of Madrid. In 2003 he became President of the Spanish Society of Construction History. From 1992 until the present he has been a consulting engineer for the restoration of many historical constructions, including the Cathedral of Tudela, San Juan de los Reyes in Toledo and the Basílica de los Desampardos among others, as well as some medieval masonry bridges. Since 1983 his research has focused on arches, vaults and domes, masonry vaulted achitecture in general. He is the author of *Arcos, bóvedas y cúulas. Geometria y equilibrio en el cálculo tradicional de estructuras de fábrica* (Madrid: Instituto Juan de Herrara, 2004).