The Black Point

Dr. Zee, the intrepid explorer, was sailing his ship through some of the unchartered regions of Flatland when his ship's sensors detected a strong force field.

"Yikes!" shrieked Dr. Zee, "look at those readings."

"We must be entering the Unspeakable Triangle," whispered his talented assistant. "The force function is

$$\mathbf{F}(x,y) = \frac{-10^5}{(x^2+y^2)^{\frac{3}{2}}}\langle x, y\rangle$$

and that could only mean..."

"A black point!" interrupted Dr. Zee.

"Yes," continued his talented assistant, "and I calculate that we are now at the point (3,2). I recommend that we lay in a course which follows the parabola with equation $x = y^2 - 1$.

"That's right," said Dr. Zee interrupting a second time, "The work done by the field as we move from (3,2) to (3,-2) will be zero since

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_C \frac{-10^5}{(x^2+y^2)^{\frac{3}{2}}}\langle x,y\rangle\cdot\langle dx, dy\rangle \\ &= -10^5\int_C \frac{1}{(x^2+y^2)^{\frac{3}{2}}}(x\,dx + y\,dy) \\ &= -10^5\int_2^{-2} \frac{1}{[(y^2-1)^2+y^2)]^{\frac{3}{2}}}\left[(y^2-1)2y\,dy + y\,dy\right] \\ &= 10^5\int_{-2}^{2} \frac{2y^3-y}{[(y^2-1)^2+y^2]^{\frac{3}{2}}}\,dy \end{aligned}$$

which is the integral of an odd function over symmetric interval."

"That's right," agreed his assistant, "but I calculate that we are now at the point $(-1, 0)$." "How much work will it take to follow the parabola and escape this thing?"

Dr. Zee answered by explaining that

$$\begin{aligned} W &= \lim_{\tau\to-\infty} -10^5\int_0^{\tau} \frac{2y^3-y}{[(y^2-1)^2+y^2]^{\frac{3}{2}}}\,dy \\ &= \lim_{\tau\to-\infty} -10^5\int_0^{\tau} \frac{2y^3-y}{[y^4-y^2-1]^{\frac{3}{2}}}\,dy. \end{aligned}$$

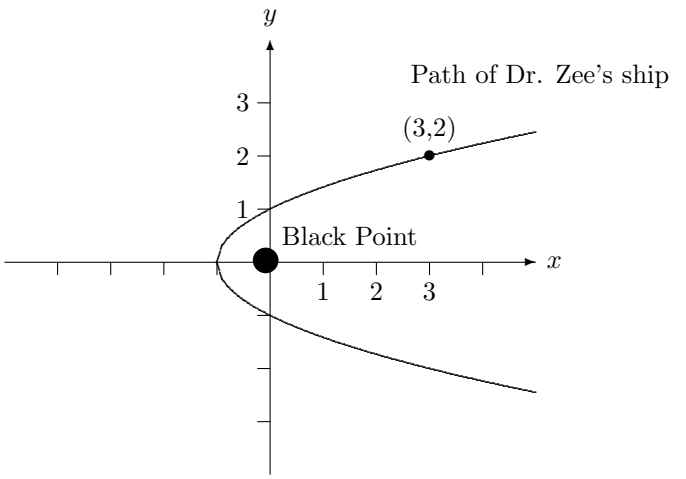
"If we set $u = y^4 - y^2 + 1$ then $du = (4y^3 - 2y)dy$ and we get

$$\begin{aligned} W &= \lim_{\tau \to -\infty} -10^5 \int_1^{\tau^4 - \tau^2 + 1} \frac{u^{-3/2}}{2}\, du \\ &= \lim_{\tau \to -\infty} -10^5 \left[-u^{-1/2} \right]_1^{\tau^4 - \tau^2 + 1} \\ &= \lim_{\tau \to -\infty} 10^5 \left((\tau^4 - \tau^2 + 1)^{-1/2} - 1 \right) \\ &= -10^5 \end{aligned}$$

"The work is negative; I've discovered antigravity!" shouted Dr. Zee.

"Wait," answered his assistant as she sat down, "W is negative since W represents the work done by the force field on our ship as we move along the charted course."

"Of course," said Dr. Zee, "move ahead at half speed, steady as she goes."



Uncharted area near the Unspeakable Triangle of Flatland

To be continued . . . . .