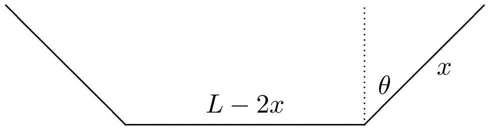
The Mystery of the Martian Canals

Dr. Zee, the intergalactically known space traveler, was amazed by the regular shape of the canals on Mars. In one of his recent lectures, he used the following diagram.



According to Dr. Zee, the angle θ has the same value in all canals and quantities x and L always have the same relationship to each other. Realizing that the Martians know very little about mathematics, Dr. Zee was puzzled.

One of his listeners reminded Dr. Zee that much evidence exists for ancient visits to Mars by other civilizations and suggested that some of these visitors may have directed the construction of the canals. At this point Dr. Zee remembered that the Artesians, who were clever with water and well schooled in mathematics, had the technology for space travel thousands of years ago.

"The mystery is solved," shouted Dr. Zee. "The Artesians surely reasoned as follows: To maximize the flow capacity we must maximize the area of the cross section. This is a function of two variables given by

$$A(\theta, x) = x\cos\theta(L - 2x + x\sin\theta)."$$

The audience sat spellbound as Dr. Zee continued. "We must find θ and x to maximize $A(\theta, x)$," shouted Dr. Zee.

The following steps can be used:

$$\begin{aligned} A(\theta, x) &= xL\cos\theta - 2x^2\cos\theta + x^2\sin\theta\cos\theta \\ &= xL\cos\theta - 2x^2\cos\theta + \frac{x^2}{2}\sin 2\theta \end{aligned}$$

Then taking partial derivatives, we get

$$\begin{aligned} A_\theta(\theta, x) &= -xL\sin\theta + 2x^2\sin\theta + x^2\cos 2\theta \\ A_{\theta\theta}(\theta, x) &= -xL\cos\theta + 2x^2\cos\theta - 2x^2\sin 2\theta \\ A_x(\theta, x) &= L\cos\theta - 4x\cos\theta + x\sin 2\theta \\ A_{xx}(\theta, x) &= -4\cos\theta + \sin 2\theta \\ A_{\theta x}(\theta, x) &= -L\sin\theta + 4x\sin\theta + 2x\cos 2\theta \end{aligned}$$

Surely any (θ, x) yielding a maximum must be a solution of the equations

$$A_\theta(\theta, x) = 0 \text{ and } A_x(\theta, x) = 0,$$

which after simplification become

$$2x + \frac{x\cos 2\theta}{\sin\theta} = L \text{ and } -4x + 2x\sin\theta = -L.$$

Adding these equations we get

$$-2x + 2x\sin\theta + \frac{x\cos 2\theta}{\sin\theta} = 0$$

which simpfies to

$$\sin\theta = \frac{1}{2}.$$

Thus $\theta = \pi/6$ and the corresponding x is $L/3$

By this time Dr. Zee was standing on the lectern shouting at his enthusiastic listeners. “Remember the test for extrema given in section 12.8 of the calculus book by Larson, Hostetler, and Edwards,” he cried.

“If $d(\theta, x) = A_{\theta\theta}(\theta, x)A_{xx}(\theta, x) - A^2_{\theta x}(\theta, x)$, then $d(\pi/6, L/3) = \frac{9L^2}{12} - \frac{L^2}{4} > 0$ and $A_{\theta\theta}(\pi/6, L/3) = \frac{-3L^2}{6} < 0$ so $A(\pi/6, L/3)$ must be a local maximum.” “From the nature of the problem $A(\pi/6, L/3)$ must also be an absolute maximum,” Dr. Zee said.

Looking at his notebook Dr. Zee concluded, “Those are precisely the numbers we found when we measured the canals on Mars. Those clever Artesians, I should have suspected them all along,” muttered Dr. Zee as he left the auditorium with thunderous applause ringing in his ears.