Dr. Zee's Diet Drink

Dr. Zee and his faithful assistant were working to install a fence that was to surround a rectangular play area for his children. They had worked for several hours in the hot sun, and they needed a break.

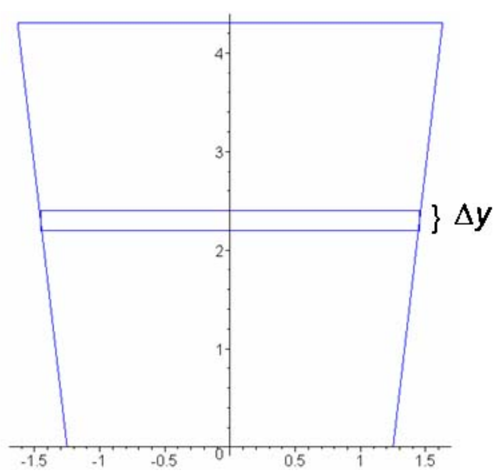
"Wow, it's hot out here," said Dr. Zee, "I could use a cold drink."

"I too," answered his talented assistant, "We have a new low calorie soft drink in the refrigerator."

The two scientists poured ice cold drinks and then sat in the shade under a tree. "I wonder how much work is done when one sucks this drink up through the straw, and how long should the straw be in order for the work to cancel the calories in the drink?"

"Well," answered his talented assistant, "These 16-oz drinking cups have radii measuring 1.25 inches at the bottom and 1.63 inches at the top, and their height measures 4.31 inches." "We'll calculate the work done in drinking one of these sodas through a straw of length L."

Dr. Zee quickly drew the following diagram in the dust under the tree:



"Here is a drawing of a side view of the cup," continued Dr. Zee. "We'll simplify the problem by subdividing the interval [0, 4.31] on vertical axis with the partition

$$0 = y_0 < y_1 < y_2 < \cdots < y_n = 4.31.$$

Let the rectangle of height Δy extend on the vertical axis from y_i to y_{i+1} and represent a typical 'slice' of this drink. We shall find the work done in raising this typical slice to the top of the straw. We are going to need the equation of the line forming right-hand side of this cup. The slope of this line is

$$m = \frac{\text{rise}}{\text{run}} = \frac{4.31 - 0}{1.63 - 1.25} \approx 11.3421,$$

and the equation is

$$y - 0 = 11.3421(x - 1.25)$$

or

$$y = 11.3421x - 14.1776.$$

We next pick $u_i \in [y_{i-1}, y_i]$ on the vertical axis inside our slice and find the radius of the cup at a height of u_i. We do this by setting $y = u_i$ in the above equation and solving for x."

Dr. Zee wrote the following equations in the dust:

$$u_i = 11.3421x - 14.1776$$

$$x = \frac{u_i + 14.1776}{11.3421}$$

$$x = 0.08816u_i + 1.25.$$

"We are now able to calculate the approximate volume of our typical slice. The volume of this cylindrical shaped object is

$$V = \pi r^2 h = \pi(0.08816u_i + 1.25)^2 \Delta y.$$

Since water weights about 62.4 pounds per cubic foot, it must weigh approximately $\frac{62.4}{12^3} \approx 0.03611$ pounds per cubic inch. The force that needs to be exerted on this slice to raise it is precisely its weight. This force is

$$f_i = \pi(0.08816u_i + 1.25)^2 \Delta y(0.03611).$$

Since this slice is raised from its height, u_i, to the top of the straw of length L through a distance of $L - u_i$ the approximate work done on this slice is

$$\Delta w_i = (\text{force})(\text{distance}) = \pi(0.08816u_i + 1.25)^2 \Delta y(0.03611)(L - u_i).$$

Adding the work done for each of these slices gives us the approximate total work done

$$\sum_{i=1}^{n} \pi(0.08816u_i + 1.25)^2(0.03611)(L - u_i)\Delta y.$$

"That is a Riemann sum," stated Dr. Zee's talented assistant. "If we let the norm of the partition approach 0 in the limit, we'll get an integral."

"That's right," answered Dr. Zee. "The total work done is the limit

$$W = \lim_{||P||\to 0} \sum_{i=1}^{n} \pi(0.08816u_i + 1.25)^2(0.03611)(L - u_i)\Delta y,$$

which is equal to the integral

$$W = \int_0^{4.31} \pi(0.08816y + 1.25)^2(0.03611)(L - y)\, dy,$$

which has value $-2.3898 + 1.0198L$. The units of work here are inch-pounds. Changing to the more common foot-pounds, we have

$$W = \frac{-2.3898 + 1.0198L}{12} = -0.1992 + 0.08498L.''$$

"Very impressive, sir!" said Dr. Zee's talented assistant. "We can calculate the work for a straw of any length L."

"Yes," said Dr. Zee. "Now we'll see how long the straw would have to be in order for the work done in sucking the drink through the straw to equal the number of calories in the drink. First, we must convert foot-pounds to Kcal, the unit we call a food 'calorie.' Doing this conversion yields

$$\begin{aligned} W &= (-0.1992 + 0.08498L) \text{ lb ft} \\ &= (-0.1992 + 0.08498L)(0.000324) \text{ Kcal} \\ &= -0.00006453 + 0.00002753L \text{ Kcal.} \end{aligned}$$

We'll first find the length of straw needed for a 200 calorie drink. Solving the equation

$$200 = -0.00006453 + 0.00002753L$$

for L leads to

$$\begin{aligned} L &= 0.7264(10^7) \text{ feet} \\ &= \frac{0.7264(10^7}{5280} \text{ miles} \\ &= 1375.7 \text{ miles.}'' \end{aligned}$$

"Holy smoke!" shouted Dr. Zee, "Let's try the same thing for a 0.5 calorie drink. Find the length of straw corresponding to 0.5 calories. Solving the equation

$$0.5 = -0.00006453 + 0.00002753L$$

for L yields

$$L = 18161 \text{ feet} = 3.4397 \text{ miles.}''$$

"That's a really long straw, sir," observed Dr. Zee's assistant.

"Yes, of course this seems not to be a good dieting method. Perhaps we could design a high viscosity diet drink with a very small straw."

With that, Dr. Zee and his faithful assistant ended their break and made their way back to work on the fence.