2D Co-rotational Truss Formulation

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Introduction

This article presents information necessary for a simple two-dimensional co-rotational truss formulation. The truss structure is allowed to have arbitrarily large displacements and rotations at the global level (so long as local truss element strains are small the results are valid). All truss elements are assumed to remain linear elastic. As with any co-rotational formulation three ingredients are required. They are (i) the relations between global and local variables, (ii) the angle of rotation of a co-rotating frame, (iii) and a variationally consistent tangent stiffness matrix. Each of theses ingredients are presented below, however, first some preliminary information is developed.

Co-rotational Concept

Let us consider the co-rotational concept in terms of truss elements. As a truss structure is loaded the entire truss deforms from its original configuration. During this process an individual element potentially does three things; it rotates, translates and deforms. The global displacements of the end nodes of the truss element include information about how the truss element has rotated, translated and deformed. The rotation and translation are rigid body motions, which may be removed from the motion of the truss. If this is done, all that remains are the strain causing deformations of the truss element. The strain causing local deformations are related to the force induced in the truss element. A co-rotational formulation seeks to separate rigid body motions from strain producing deformations at the local element level. This is accomplished by attaching a local element reference frame (or coordinate system), which rotates and translates with the truss element. For 2D trusses this amounts to attaching a co-rotating frame with origin at node one of the truss such that the x-axis is always directed along the truss element. The y-axis is perpendicular to the x-axis so that the result is a right handed orthogonal coordinate system. With respect to this local co-rotating coordinate frame the rigid body rotations and translations are zero and only local strain producing deformations along the x-axis remain.

Preliminary Information

Consider a typical truss member in its initial and current configurations as shown in Figure 1. For the truss member in its initial configuration the global nodal coordinates are defined as (X_1, Y_1) for node 1 and (X_2, Y_2) for node 2. The original length of the truss member is

$$L_o = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}. \tag{1}$$

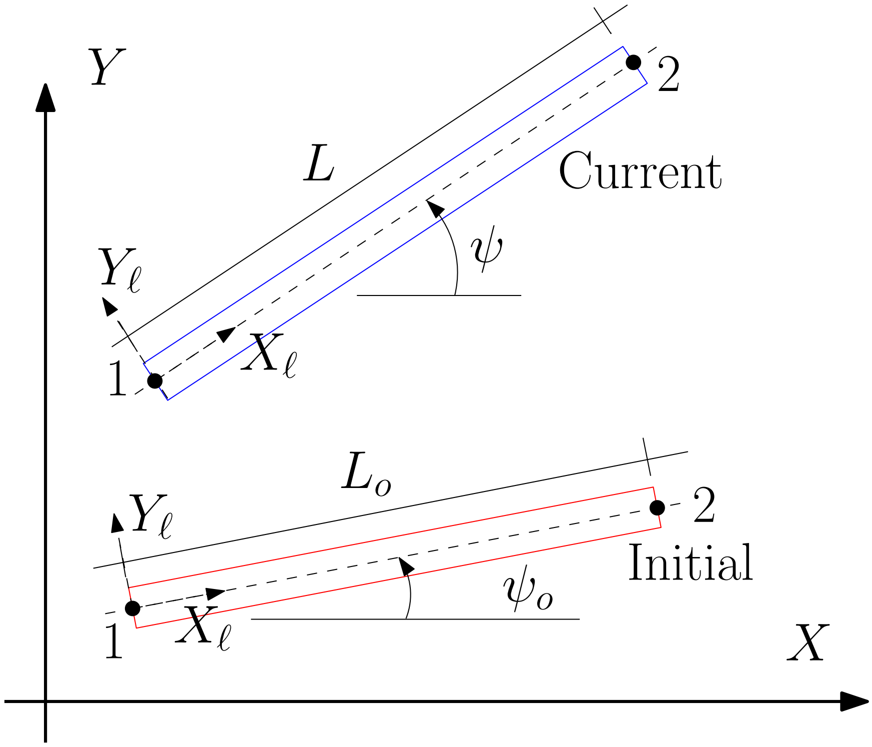


Figure 1: Initial and current configuration for typical truss member.

For the truss member in its current configuration the global nodal coordinates are $(X_1 + v_{X1}, Y_1 + v_{Y1})$ for node 1 and $(X_2 + v_{X2}, Y_2 + v_{Y2})$ for node 2, where for example, v_{X1} is the global nodal displacement of node 1 in the X direction. The current length of the truss member is

$$L = \sqrt{((X_2 + v_{X2}) - (X_1 + v_{X1}))^2 + ((Y_2 + v_{Y2}) - (Y_1 + v_{Y1}))^2}. \tag{2}$$

Relation between global and local variables

Relations between global and local variables are necessary to calculate the local axial deformation for a truss member. The axial deformation is used to calculate the internal force of the truss member in local coordinates. The axial deformation of the truss member, d, is calculated as

$$d = L - L_o. \tag{3}$$

Note that the L and L_o are calculated in terms of global variables (the initial coordinates and the current nodal displacements). In the case of a truss element these relations that allow extraction of the local deformation are quite simple. However, in the case of beams, continua and shells, the relations are not necessarily so simple. In addition to the above relations it is also necessary to find relations between local and global axial forces. Such relations come about naturally in the process of finding the consistent tangent stiffness matrix and hence are left until that section.

Angle of rotation of the co-rotating frame

The global coordinates remain fixed throughout the co-rotational formulation. However, a local co-rotating coordinate frame is attached to each truss member as shown in Figure 1. This co-rotating coordinate frame rotates with the truss member as the truss structure deforms. The current angle of the co-rotating frame with respect to the global coordinate system is denoted as ψ. In a two-dimensional truss formulation it is helpful to calculate the current sine and cosine values of this angle. This is accomplished with the following expressions which use global variables.

$$\cos\psi = \frac{(X_2 + v_{X2}) - (X_1 + v_{X1})}{L}, \quad \sin\psi = \frac{(Y_2 + v_{Y2}) - (Y_1 + v_{Y1})}{L} \tag{4}$$

Variationally consistent tangent stiffness matrix

To find the variationally consistent tangent stiffness matrix recall that the potential energy, U, of a spring is written as follows:

$$U(d) = \frac{1}{2}kd^2 \tag{5}$$

In this case d is the axial displacement of the spring (or truss bar) and k is the spring (or truss bar) stiffness. For the case of a truss bar the stiffness is AE/L_o, where A is the cross-sectional area of the truss bar and E is the modulus of elasticity of the bar material. Next, define a vector of global nodal displacements for a typical truss member as

$$\mathbf{v} = \left\{ \begin{array}{c} v_{X1} \\ v_{Y1} \\ v_{X2} \\ v_{Y2} \end{array} \right\}. \tag{6}$$

The internal force vector $\mathbf{f}_g$ at the global level is found by taking the derivative of the potential energy with respect to the vector of global nodal displacements. That is

$$\mathbf{f}_g = \frac{\partial U}{\partial \mathbf{v}} = kd \begin{bmatrix} \frac{\partial d}{\partial v_{X1}} \\ \frac{\partial d}{\partial v_{Y1}} \\ \frac{\partial d}{\partial v_{X2}} \\ \frac{\partial d}{\partial v_{Y2}} \end{bmatrix}. \tag{7}$$

Now define the axial force in the truss member as

$$N = kd = \frac{AE}{L_o}d. \tag{8}$$

To find the partial derivatives in (7) observe that for example

$$\frac{\partial d}{\partial v_{X1}} = \frac{\partial (L - Lo)}{\partial v_{X1}} = \frac{\partial L}{\partial v_{X1}}. \tag{9}$$

Then by using (2) it can be shown that

$$\frac{\partial L}{\partial v_{X1}} = -\cos\psi, \quad \frac{\partial L}{\partial v_{Y1}} = -\sin\psi \tag{10}$$

$$\frac{\partial L}{\partial v_{X2}} = \cos\psi, \quad \frac{\partial L}{\partial v_{Y2}} = \sin\psi. \tag{11}$$

Hence, using these last expressions in (7) gives

$$\mathbf{f}_g = N \begin{bmatrix} -\cos\psi \\ -\sin\psi \\ \cos\psi \\ \sin\psi \end{bmatrix}. \tag{12}$$

This last relation expresses the global force components for a single truss member as a function of the local internal axial force, N. This is an important relationship for constructing the internal force vector in global coordinates. In traditional truss formulations the relationship between global and local forces is sometimes written as

$$\mathbf{f}_g = \mathbf{T}^T\mathbf{f} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}^T \begin{Bmatrix} -N \\ 0 \\ N \\ 0 \end{Bmatrix}, \tag{13}$$

where the result is identical to (12) and $\mathbf{T}$ is the traditional transformation matrix.

The variationally consistent tangent stiffness is found by taking the second derivative of the potential energy with respect to the global nodal displacement vector as follows:

$$\frac{\partial^2 U}{\partial \mathbf{v}^2} = \frac{\partial \mathbf{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}(kd\frac{\partial d}{\partial \mathbf{v}}) = k\frac{\partial d}{\mathbf{v}}\left(\frac{\partial d}{\mathbf{v}}\right)^T + kd\frac{\partial^2 d}{\mathbf{v}^2} = \mathbf{K}_M + \mathbf{K}_G. \tag{14}$$

The tangent stiffness matrix is composed of two terms, the material stiffness, $\mathbf{K}_M$, and the geometric stiffness, $\mathbf{K}_G$. Substituting into the expression for the material stiffness gives

$$\mathbf{K}_M = \frac{AE}{L_o} \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix} [-c \quad -s \quad c \quad s] \tag{15}$$

where the column vector and row vector are multiplied together as a tensor product which results in the following standard expression for the global stiffness of a typical truss member.

$$\mathbf{K}_M = \frac{AE}{L_o} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} = \mathbf{T}^T\mathbf{k}\mathbf{T} \tag{16}$$

where $\mathbf{T}$ is the standard transformation matrix given previously and $\mathbf{k}$ is the local truss stiffness matrix given as

$$\mathbf{k} = \frac{AE}{L_o}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

To find the geometric stiffness matrix the second derivatives of d with respect to $\mathbf{v}$ must be determined. Noting that

$$\frac{\partial^2 d}{\partial \mathbf{v}^2} = \frac{\partial^2 L}{\partial \mathbf{v}^2} \tag{18}$$

it then can be shown that

$$\frac{\partial^2 L}{\partial v_{X1}^2} = \frac{s^2}{L}, \quad \frac{\partial^2 L}{\partial v_{Y1}^2} = \frac{c^2}{L}, \quad \frac{\partial^2 L}{\partial v_{X1}\partial v_{Y1}} = \frac{-cs}{L}, \quad etc. \tag{19}$$

so that

$$\mathbf{K}_G = \frac{N}{L}\begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}. \tag{20}$$

The last expression for $\mathbf{K}_G$ is equivalent to

$$\mathbf{K}_G = \mathbf{T}^T\left(\frac{N}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right)\mathbf{T}. \tag{21}$$

The matrix expression in parenthesis comes about by considering geometric effects in a small strain setting (see Cook and Malkus...). By considering geometric effects in a large strain setting the traditional geometric stiffness is obtained as

$$\mathbf{K}_G = \mathbf{T}^T\left(\frac{N}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\right)\mathbf{T} = \frac{N}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{22}$$

where the final expression comes about due to the fact that the geometric stiffness is invariant with respect to rotations (ie, $\mathbf{K}_G = \mathbf{T}^T\mathbf{K}_G\mathbf{T}$) and is therefore identical in the local and global coordinate system.

For an alternative derivation the reader is directed to the work by Crisfield.

Load Control Algorithm for Co-rotational Truss Analysis

The following is a load control algorithm for performing a co-rotational truss analysis. This is an implicit formulation which uses Newton-Raphson iterations at the global level to achieve equilibrium during each incremental load step. Material nonlinearities are not presently included in the algorithm. A program implementing this algorithm has been written in MATLAB and some representative results are provided. The algorithm proceeds as follows:

1. Define/initialize variables

   - $\mathbf{F}$ = the total vector of externally applied global nodal forces
   - $\mathbf{F}^{n+1}$ = the current externally applied global nodal force vector
   - $\mathbf{N}$ = the vector of truss axial forces, axial force in truss element i is N_i
   - $\mathbf{u}$ = the vector of global nodal displacements, initially $\mathbf{u} = \mathbf{0}$
   - $\mathbf{x}$ = the vector of nodal x coordinates in the undeformed configuration
   - $\mathbf{y}$ = the vector of nodal y coordinates in the undeformed configuration
   - $\mathbf{L}$ = the vector of truss element lengths based on current $\mathbf{u}$ using equation (2), L for truss i is L_i, save the initial lengths in a vector $\mathbf{L}_o$ by using equation (1).
   - $\mathbf{c}$ and $\mathbf{s}$ = the vectors of cosines and sines for each truss element angle based on the current $\mathbf{u}$ using equations (4).
   - $\mathbf{K} = \mathbf{K_M} + \mathbf{K_G}$, the assembled global tangent stiffness matrix
   - $\mathbf{K}_s$ = the modified global tangent stiffness matrix to account for supports. Rows and columns associated with zero displacement dofs are set to zero and the diagonal position is set to 1.

2. **Start Loop** over load increments (for $n = 0$ to $ninc - 1$).

   (a) Calculate load factor $\lambda = 1/ninc$ and incremental force vector $d\mathbf{F} = \lambda\mathbf{F}$.

   (b) Calculate global stiffness matrix $\mathbf{K}$ based on current values of $\mathbf{c}, \mathbf{s}, \mathbf{L}$ and $\mathbf{N}$.

   (c) Modify $\mathbf{K}$ to account for supports and get $\mathbf{K}_s$.

   (d) Solve for the incremental global nodal displacements $d\mathbf{u} = \mathbf{K}_s^{-1} d\mathbf{F}$

   (e) Update global nodal displacements, $\mathbf{u}^{n+1} = \mathbf{u}^n + d\mathbf{u}$

   (f) Update the global nodal forces, $\mathbf{F}^{n+1} = \mathbf{F}^n + d\mathbf{F}$

   (g) Update $\mathbf{L}$,$\mathbf{c}$ and $\mathbf{s}$

   (h) Calculate the vector of new internal truss element axial forces $\mathbf{N}^{n+1}$. For truss element i the axial force is $N_i^{n+1} = (A_i E / L_{oi}) [L_i - L_{oi}]$.

   (i) Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}^{n+1}$.

   (j) Calculate the residual $\mathbf{R} = \mathbf{F}_{int}^{n+1} - \mathbf{F}^{n+1}$ and modify the residual to account for the required supports.

   (k) Calculate the norm of the residual $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

   (l) Iterate for equilibrium if necessary. Set up iteration variables.

      - Iteration variable = $k = 0$
      - $tolerance = 10^{-6}$
      - $maxiter = 100$
      - $\delta\mathbf{u} = \mathbf{0}$
      - $\mathbf{N}_{temp} = \mathbf{N}^{n+1}$

(m) **Start Iterations** while $R > tolerance$ and $k < maxiter$

i. $\mathbf{N}_{temp} = \mathbf{N}^{n+1}$

ii. Calculate the new global stiffness $\mathbf{K}$

iii. Modify the global stiffness to account for supports which gives $\mathbf{K}_s$

iv. Calculate the correction to $\mathbf{u}^{n+1}$, which is $\delta\mathbf{u}^{k+1} = \delta\mathbf{u}^k - \mathbf{K}_s^{-1}\mathbf{R}$, but note that $\mathbf{u}^{n+1}$ is not updated until all iterations are completed

v. Update $\mathbf{L}$,$\mathbf{c}$ and $\mathbf{s}$ based on current $\mathbf{u}^{n+1} + \delta\mathbf{u}^{k+1}$

vi. Calculate the vector of new internal truss element axial forces $\mathbf{N}_{temp}^{k+1}$. For truss element i the axial force is $(N_{temp}^{k+1})_i = (A_i E/L_{oi})\,[L_i - L_{oi}]$.

vii. Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}_{temp}^{k+1}$.

viii. Calculate the residual $\mathbf{R} = \mathbf{F}_{int}^{n+1} - \mathbf{F}^{n+1}$ and modify the residual to account for the required supports.

ix. $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

x. Update iterations counter $k = k + 1$

(n) **End** of while loop iterations

3. Update variables to their final value for the current increment

- $\mathbf{N}^{n+1} = \mathbf{N}_{temp}$
- $\mathbf{u}_{final}^{n+1} = \mathbf{u}_{(0)}^{n+1} + \delta\mathbf{u}_{(k)}$

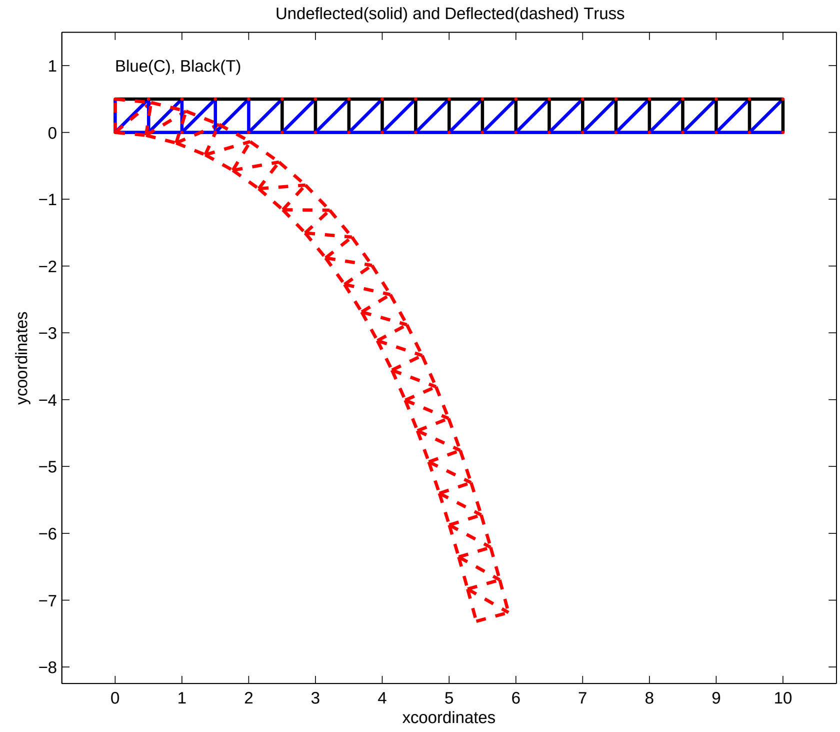
4. **End Loop** over load increments

Load Control Example Results

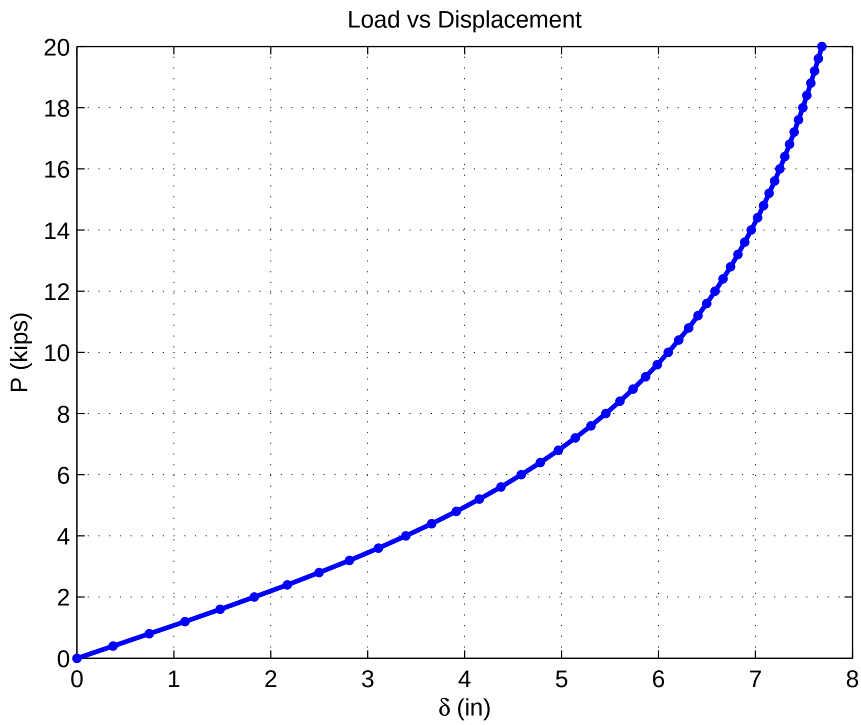
A cantilever truss is loaded by a point load at its free end. The initial and final configuration of the truss is shown in Figure 2a. The load displacement results are shown in the plot of Figure 2b. The load displacement is linear for small loads, but as the load increases the curve is clearly nonlinear. As the load increases further the structure becomes stiffer, which is caused by tension stiffening of the truss in its deformed configuration. The analysis is completed in 50 equal load increments. For this example, all truss members have a cross-sectional area, $A = 0.1$ in^2, and modulus of elasticity, $E = 29000$ ksi. The truss is 10 inches long with 0.5 inch long vertical members and 0.5 inch long horizontal members. As a result of the above dimensions there are 42 nodes and 81 members. The tolerance used for equilibrium iterations is 10^{-6}.

Co-rotational algorithm – Displacement control

The following implicit algorithm uses Newton-Raphson iterations within each specified displacement increment to enforce global equilibrium for the truss structure (see Clarke and Hancock for displacement control details). The specified displacement increments are prescribed at a structure dof chosen by the user. Typically this is the structure dof of maximum displacement in the dof direction. Here equal size displacement increments are used. The algorithm proceeds as follows:



(a)



(b)

Figure 2: Cantilever Truss Analyzed By Load Control: (a) Truss deflected shape, (b) Load versus displacement plot.

1. Define/initialize variables

   - D_{max} = the user specified maximum displacement at dof q
   - $ninc$ = the user specified number of displacement increments to reach D_{max}
   - $\Delta\bar{u}_q = D_{max}/ninc$ = the specified incremental displacement at dof q
   - $\mathbf{F}$ = the total vector of externally applied global nodal forces
   - $\mathbf{F}^{n+1}$ = the current externally applied global nodal force vector
   - λ^{n+1} = the current load ratio, that is $\lambda^{n+1}\mathbf{F} = \mathbf{F}^n + d\mathbf{F} = \mathbf{F}^n + d\lambda^{n+1}\mathbf{F} = \mathbf{F}^{n+1}$, the load ratio starts out equal to zero
   - $\mathbf{N}$ = the vector of truss axial forces, axial force in truss element i is N_i
   - $\mathbf{u}$ = the vector of global nodal displacements, initially $\mathbf{u} = \mathbf{0}$
   - $\mathbf{x}$ = the vector of nodal x coordinates in the undeformed configuration
   - $\mathbf{y}$ = the vector of nodal y coordinates in the undeformed configuration
   - $\mathbf{L}$ = the vector of truss element lengths based on current $\mathbf{u}$ using equation (2), L for truss i is L_i, save the initial lengths in a vector $\mathbf{L}_o$ using equation (1)
   - $\mathbf{c}$ and $\mathbf{s}$ = the vectors of cosines and sines for each truss element angle based on the current $\mathbf{u}$ using equations (4).
   - $\mathbf{K} = \mathbf{K_M} + \mathbf{K_G}$, the assembled global tangent stiffness matrix
   - $\mathbf{K}_s$ = the modified global tangent stiffness matrix to account for supports. Rows and columns associated with zero displacement dofs are set to zero and the diagonal position is set to 1.

2. **Start Loop** over load increments (for $n = 0$ to $ninc - 1$).

   (a) Calculate global stiffness matrix $\mathbf{K}$ based on current values of $\mathbf{c}, \mathbf{s}, \mathbf{L}$ and $\mathbf{N}$.

   (b) Modify $\mathbf{K}$ to account for supports and get $\mathbf{K}_s$.

   (c) Calculate the incremental load ratio $d\lambda^{n+1}$. The incremental load ratio is calculated as follows. Calculate a displacement vector based on the current stiffness, that is $\hat{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{F}$. Take from $\hat{\mathbf{u}}$ the displacement in the direction of dof q, that is $\hat{u}_q$. Then $d\lambda^{n+1} = \Delta\bar{u}_q/\hat{u}_q$. Update load ratio $\lambda^{n+1} = \lambda^n + d\lambda^{n+1}$.

   (d) Calculate the incremental force vector $d\mathbf{F} = d\lambda^{n+1}\mathbf{F}$.

   (e) Solve for the incremental global nodal displacements $d\mathbf{u} = \mathbf{K}_s^{-1}d\mathbf{F}$

   (f) Update global nodal displacements, $\mathbf{u}^{n+1} = \mathbf{u}^n + d\mathbf{u}$

   (g) Update $\mathbf{L}$,$\mathbf{c}$ and $\mathbf{s}$

   (h) Calculate the vector of new internal truss element axial forces $\mathbf{N}^{n+1}$. For truss element i the axial force is $N_i^{n+1} = (A_iE/L_{oi})\,[L_i - L_{oi}]$.

   (i) Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}^{n+1}$.

   (j) Calculate the residual $\mathbf{R} = \lambda^{n+1}\mathbf{F} - \mathbf{F}_{int}^{n+1}$ and modify the residual to account for the required supports.

(k) Calculate the norm of the residual $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

(l) Iterate for equilibrium if necessary. Set up iteration variables.

- Iteration variable $= k = 0$
- $tolerance = 10^{-6}$
- $maxiter = 100$
- $\delta \mathbf{u} = \mathbf{0}$
- $\delta \lambda = 0$
- $\mathbf{N}_{temp} = \mathbf{N}^{n+1}$

(m) **Start Iterations** while $R > tolerance$ and $k < maxiter$

i. $\mathbf{N}_{temp} = \mathbf{N}^{n+1}$

ii. Calculate the new global stiffness $\mathbf{K}$

iii. Modify the global stiffness to account for supports which gives $\mathbf{K}_s$

iv. Calculate the load ratio correction $\delta\lambda_{k+1}$. The load ratio correction is calculated as follows. Calculate $\breve{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{R}$ and $\hat{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{F}$. From $\breve{\mathbf{u}}$ and $\hat{\mathbf{u}}$ extract the component of displacement in the direction of dof q, that is $\breve{u}_q$ and $\hat{u}_q$. Then $\delta\lambda_{k+1} = \delta\lambda_k - \breve{u}_q/\hat{u}_q$.

v. Calculate the correction to $\mathbf{u}^{n+1}$, which is $\delta\mathbf{u}^{k+1} = \delta\mathbf{u}^k + \mathbf{K}_s^{-1}\left[\mathbf{R} - (\breve{u}_q/\hat{u}_q)\mathbf{F}\right]$, but note that $\mathbf{u}^{n+1}$ is not updated until all iterations are completed

vi. Update $\mathbf{L}$,$\mathbf{c}$ and $\mathbf{s}$ based on current $\mathbf{u}^{n+1} + \delta\mathbf{u}^{k+1}$

vii. Calculate the vector of new internal truss element axial forces $\mathbf{N}_{temp}^{k+1}$. For truss element i the axial force is $(N_{temp}^{k+1})_i = (A_i E/L_{oi})\left[L_i - L_{oi}\right]$.

viii. Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}_{temp}^{k+1}$.

ix. Calculate the residual $\mathbf{R} = (\lambda^{n+1} + \delta\lambda_{k+1})\mathbf{F} - \mathbf{F}_{int}^{n+1}$ and modify the residual to account for the required supports.

x. $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

xi. Update iterations counter $k = k + 1$

(n) **End** of while loop iterations

3. Update variables to their final value for the current increment

- $\lambda_{final}^{n+1} = \lambda_{(0)}^{n+1} + \delta\lambda_k$
- $\mathbf{N}^{n+1} = \mathbf{N}_{temp}$
- $\mathbf{u}_{final}^{n+1} = \mathbf{u}_{(0)}^{n+1} + \delta\mathbf{u}_{(k)}$

4. **End Loop** over load increments

Displacement Control Example Results

A parabolic arch truss is loaded by a point load at midspan. The initial and final configuration of the truss is shown in Figure 3a. The load displacement results are shown in the plot of Figure 3b. The phenomenon of snap through buckling is illustrated in the load displacement results. The displacement control algorithm successfully traces the load displacement path, whereas using the load control algorithm would not be able to correctly trace the whole path. The analysis is complete in 25 equal sized displacement increments in order to reach a total midspan deflection of 2.5 inches downwards.

Control Schemes

The examples mentioned previously are for load control and displacement control. These techniques will work for many practical problems. However, for problems of snap back and equilibrium path tracing (see Figure 4) other methods such as generalized displacement control (see Yang et al) or arc length control (see Crisfield or Clarke and Hancock) will be necessary. An excellent summary of many of the methods is given in the work by McGuire et al.

Some Implementation Details–Calculating internal force vector

In the algorithms above the vector **N** is just a temporary vector to store the current axial force in each member of the truss. Hence, for a truss structure with n_m members the vector **N** is n_m by 1 in size. Using, the ith row from **N** the internal force vector in global coordinates for truss member i is (using equation (12))

$$\mathbf{f}_g = \mathbf{f}^i_{int} = N_i \mathbf{t}_i = N_i \begin{bmatrix} -c_i \\ -s_i \\ c_i \\ s_i \end{bmatrix}. \tag{23}$$

Then by an appropriate assembly procedure, based on truss member degrees of freedom, the global internal force vector, $\mathbf{F}_{int}$ (size $2n_{nodes} \times 1$) is constructed. That is

$$\mathbf{F}_{int} = \mathop{\mathsf{A}}_{i=1}^{n_m} \mathbf{f}^i_{int}, \tag{24}$$

where A is the assembly operator and n_m is the number of truss members in the structure.

Example – Load control numerical results for comparison

If the reader tries to implement the above load control procedure in a truss analysis program the following example problem may be used as a check for correctness of results. The results are obtained using a co-rotational truss analysis program written in MATLAB.

Consider the single bar truss structure shown in Figure 5a. The model data is as follows. The coordinates of node 1 are (0,0) and the coordinates of node 2 are (2499.875 mm, 25 mm). The bar is in the undeformed configuration with an initial length of L = 2500 mm. The initial height of the bar at node 2 is z = 25 mm. The modulus of elasticity E = 5 x 10^7 N/mm^2. The cross-sectional area of the bar is $A = 1$ mm^2. The applied load, P, is positive upwards and displacement, δ, is also positive upwards as shown.

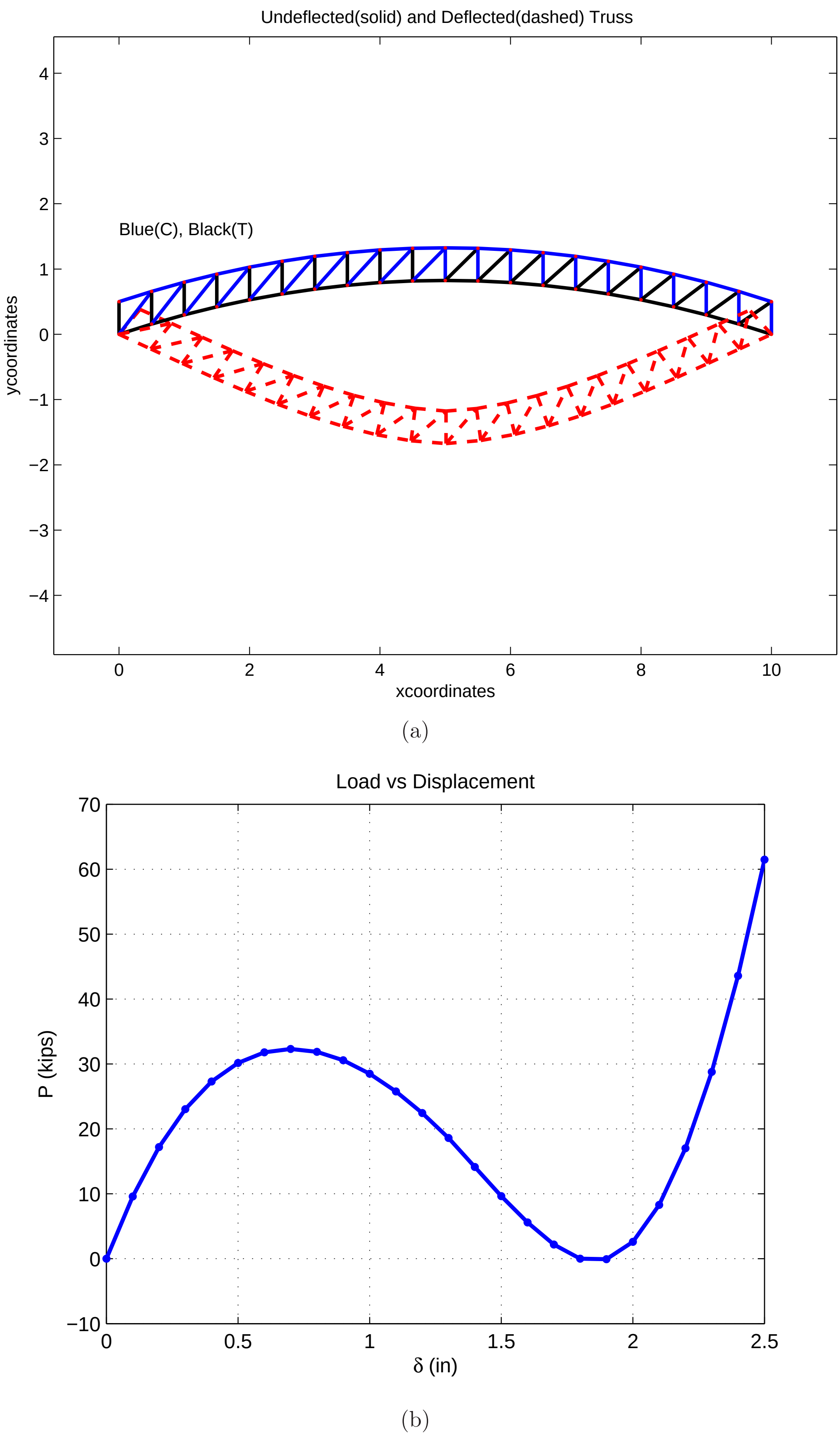
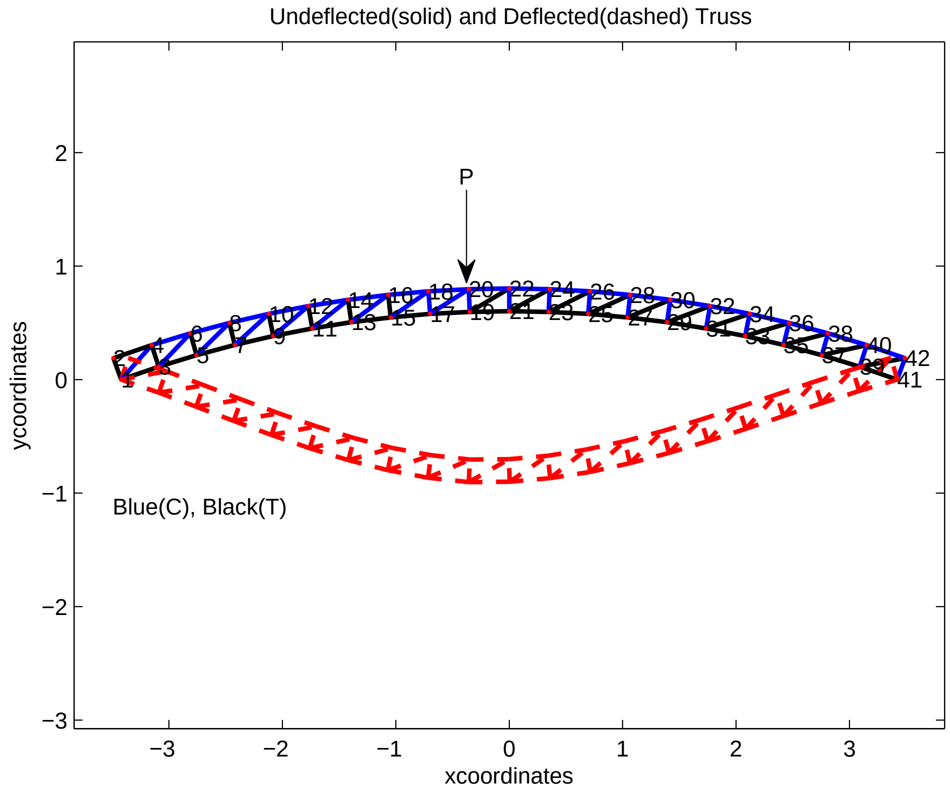
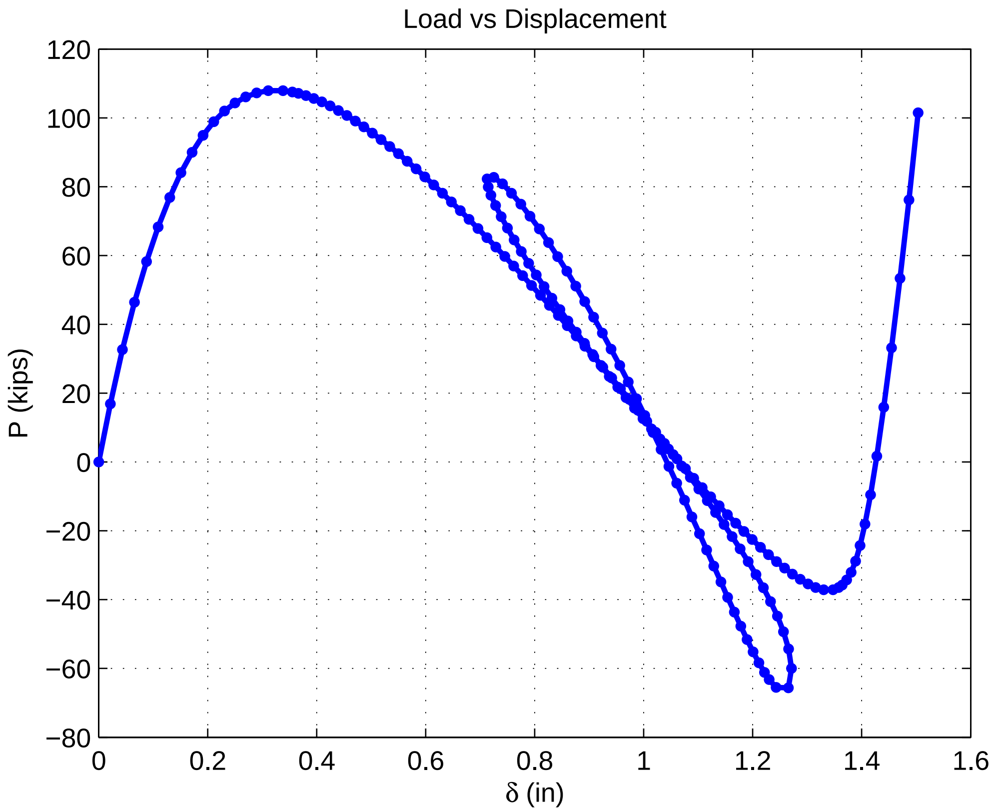


Figure 3: Parabolic Arch Truss With Midspan Point Load Analyzed By Displacement Control: (a) Truss deflected shape, (b) Load versus displacement plot.



(a)



(b)

Figure 4: Circular Arch Truss Analyzed By Generalized Displacement Control With Offset Point Load At Node 20: (a) Truss deflected shape, (b) Load versus displacement plot.

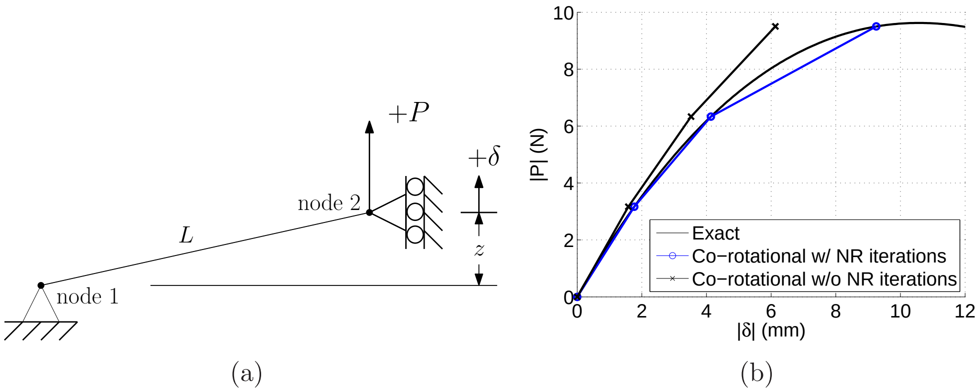


Figure 5: Single bar with one degree of freedom: (a) single bar truss structure in undeformed configuration; (b) results for downward applied load P as a nonlinear function of displacement δ (absolute values of load and displacement are plotted).

If a total load of -9.5 N (downwards) is applied to node 2 using load control in three equal load increments the graph of Figure 5b is obtained, where the absolute values of load and displacement are plotted. Results are shown for three cases, (i) the exact solution, (ii) a co-rotational truss solution with Newton-Raphson iterations and (iii) a co-rotational truss solution without Newton-Raphson iterations. The numerical results, for the three cases, are given in Table 1.

For the solution using Newton-Raphson iterations the tolerance required for equilibrium was set at 10^{-6}, and the number of iterations required for equilibrium were 3, 3 and 11 for load steps 1, 2 and 3 respectively. It is evident from Figure 5b that the solution drifts from the exact solution when no Newton-Raphson iterations are used to achieve equilibrium. However, by including Newton-Raphson iterations the numerical results closely match the exact solution.

The exact solution for P as a function of δ is given by Crisfield and is expressed as follows.

$$P(\delta) = \frac{EA}{L^3}(z^2\delta + \frac{3}{2}z\delta^2 + \frac{1}{2}\delta^3). \tag{25}$$

The equation for $P(\delta)$ is used to plot the exact solution in Figure 5b.

It is worth noting that if the solution is plotted for higher displacement values the phenomenon of snap through (like Figure 3b) is observed. Hence, the exact solution given above is helpful for verifying results for a co-rotational truss formulation using displacement control, arc length control or generalized displacement control.

Load P (N)	δ_{exact} (mm)	δ_{NR} (mm)	δ_{noNR} (mm)
-3.1667	-1.7660	-1.76605	-1.58333
-6.3333	-4.1367	-4.1367	-3.52362
-9.5000	-9.2539	-9.25387	-6.13211

Table 1: Load and displacement at node 2.

Conclusion

A derivation and explanation of the ingredients of a 2D co-rotational truss formulation, in a small strain setting, is provided. An algorithm for load control is provided with an example figure. An algorithm for displacement control is also provided with an example figure. Arc length and generalized displacement control are also identified as possible control schemes. Last, an example is provided with numerical results for comparison.

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