Nonlinear Static - 1D Plasticity - Isotropic and Kinematic Hardening
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1 Introduction

From a computational point of view hardening is provided either by isotropic hardening, kinematic hardening or by some combination of the two. It is observed from real life experiments with metals that kinematic hardening takes place when metals are loaded past the yield point. This is often called the Bauschinger effect. The Bauschinger effect essentially causes the center of the yield surface to move relative to the stress space origin when plastic flow takes place. Hence, unlike isotropic hardening, kinematic hardening can result in a work hardened material that has a different yield stress magnitude in tension than the yield stress magnitude in compression. Most polycrystalline metal materials, for example, do exhibit this kind of behavior. It is assumed that the reader has read the content contained in [8]. The current article is an extension of 1D plasticity with isotropic hardening [8] and provides additional theory and details necessary to include kinematic hardening.

2 Definitions

Brief definitions of the most important terminology are as follows. For kinematic hardening a hardening modulus, H, is used to characterize the rate of hardening per unit plastic flow. This is determined from experimentation and in a finite element implementation the hardening modulus is provided as input. To keep track of the center of the yield surface an internal variable, q, called back stress is created. Otherwise, the definitions of all variables are the same as those presented in [8] for 1D plasticity with isotropic hardening.

3 1D Plasticity w/ Isotropic and Kinematic Hardening

The algorithmic pieces of one dimensional plasticity with a general expression for isotropic and kinematic hardening is presented next. The derivations and notation closely follows that given by Simo and Hughes[6].

3.1 Derivation of Elasto-Plastic Tangent Modulus

As usual the modulus of elasticity is E, the equivalent plastic strain is α and the total strain is defined as

$$\varepsilon = \varepsilon^p + \varepsilon^e. \tag{3.1}$$

Stress is linear elastic when $f < 0$ and is calculated as

$$\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p) \tag{3.2}$$

the flow rule and back stress rate (simple Ziegler's rule), respectively, are assumed as

$$\dot{\varepsilon}^p = \gamma sign(\sigma - q). \tag{3.3}$$

$$\dot{q} = H\dot{\varepsilon}^p = \gamma H sign(\sigma - q). \tag{3.4}$$

The yield condition is defined as follows:

$$f(\sigma) = |\sigma - q| - G(\alpha) \tag{3.5}$$

,where $G(\alpha)$ is a yield stress function (see [8]) which includes the type of isotropic hardening and is a function of α. The customary Kuhn-Tucker conditions apply ($\gamma \geq 0$, $f(\sigma) \leq 0$, and $\gamma f(\sigma) = 0$). If $f(\sigma)$ is to be zero the consistency condition requires that $\gamma \dot{f}(\sigma) = 0$. Hence, when $\gamma > 0$, $\dot{f} = 0$ so that by the chain rule

$$\dot{f} = \frac{\partial f}{\partial \sigma}\frac{\partial \sigma}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial f}{\partial G}\frac{\partial G}{\partial \alpha}\frac{\partial \alpha}{\partial t} = 0. \tag{3.6}$$

Inserting each partial derivative into the above formula, simplifying and using the relations $\dot{\alpha} = \gamma$, $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$, $\dot{\varepsilon}^p = \gamma sign(\sigma)$, $\dot{q} = \gamma H sign(\sigma - q)$, $(sign(\sigma))^2 = 1$, yields

$$\begin{aligned}
\dot{f} &= sign(\sigma - q)\dot{\sigma} - sign(\sigma - q)\dot{q} - \frac{\partial G}{\partial \alpha}\dot{\alpha} \\
&= sign(\sigma - q)[E(\dot{\varepsilon} - \dot{\varepsilon}^p) - \dot{q}] - \frac{\partial G}{\partial \alpha}\dot{\alpha} \\
&= sign(\sigma - q)E\dot{\varepsilon} - sign(\sigma - q)E\dot{\varepsilon}^p - sign(\sigma - q)\dot{q} - \frac{\partial G}{\partial \alpha}\dot{\alpha} \\
&= sign(\sigma - q)E\dot{\varepsilon} - sign(\sigma - q)E\gamma sign(\sigma - q) - sign(\sigma - q)\gamma H sign(\sigma - q) - \frac{\partial G}{\partial \alpha}\gamma \\
&= sign(\sigma - q)E\dot{\varepsilon} - \gamma E - \gamma H - \frac{\partial G}{\partial \alpha}\gamma \\
&= sign(\sigma - q)E\dot{\varepsilon} - \gamma[E + H + \frac{\partial G}{\partial \alpha}] \leq 0.
\end{aligned} \tag{3.7}$$

The above result holds because $\dot{f} > 0$ is not allowed. Then, if yielding ($f = 0$) is taking place consistency requires $\dot{f} = 0$. Solving for γ using (3.7) with $\dot{f}$ set equal to zero gives

$$\gamma = \frac{sign(\sigma - q)E\dot{\varepsilon}}{E + H + \frac{\partial G}{\partial \alpha}}. \tag{3.8}$$

With the above results in hand the elasto-plastic tangent modulus, $\mathsf{C}_{ep} = \frac{d\sigma}{d\varepsilon}$, is found as follows. Observe that $\dot{\sigma} = \frac{d\sigma}{d\varepsilon}\dot{\varepsilon}$. Then using (3.2), (3.3) and (3.8) yields

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) = E(\dot{\varepsilon} - \gamma sign(\sigma - q)) = E(\dot{\varepsilon} - \frac{(sign(\sigma - q))^2 E\dot{\varepsilon}}{E + H + \frac{\partial G}{\partial \alpha}}). \tag{3.9}$$

Upon simplification the stress rate becomes

$$\dot{\sigma} = \frac{E(H + \frac{\partial G}{\partial \alpha})}{E + H + \frac{\partial G}{\partial \alpha}}\dot{\varepsilon} = \mathsf{C}_{ep}\dot{\varepsilon}. \tag{3.10}$$

By inspection of (3.10) the elasto-plastic tangent modulus is

$$\mathsf{C}_{ep} = \frac{E(H + \frac{\partial G}{\partial \alpha})}{E + H + \frac{\partial G}{\partial \alpha}}. \tag{3.11}$$

3.2 Development of algorithmic ingredients for 1D plasticity with isotropic & kinematic hardening

Consider now the algorithmic ingredients for a 1D plasticity problem. Suppose that a strain increment is provided so that a new total strain, ε, is given. From this information a trial value of the yield condition, f_{n+1}^{trial}, is calculated. If $f_{n+1}^{trial} \leq 0$ the strain is elastic and the solution is trivial. If on the other hand $f_{n+1}^{trial} > 0$ then a plastic step has occurred. For a plastic step, the problem is to find σ_{n+1}, α_{n+1}, q_{n+1} such that $f(\sigma_{n+1}, \alpha_{n+1}, q_{n+1}) = 0$ and $\Delta\gamma > 0$. Having this information then allows calculation of ε_{n+1}^p, which is the new total plastic strain. To derive an algorithm for a typical plastic step first express σ_{n+1} as a function of σ_{n+1}^{trial} and $\Delta\gamma$ as follows:

$$\begin{aligned} \sigma_{n+1} &= E(\varepsilon_{n+1} - \varepsilon_{n+1}^p) \\ &= E(\varepsilon_{n+1} - \varepsilon_n^p) - E(\varepsilon_{n+1}^p - \varepsilon_n^p) \\ &= E(\varepsilon_{n+1} - \varepsilon_n^p) - E\dot{\varepsilon}_{n+1}^p \\ &= \sigma_{n+1}^{trial} - E\Delta\gamma sign(\sigma_{n+1}). \end{aligned} \tag{3.12}$$

Assuming that the correct value of $\Delta\gamma > 0$ can be found for the current plastic step and using the relation $sign(\sigma_{n+1} - q_{n+1}) = sign(\xi_{n+1})$, here ξ is the relative stress, and all the quantities are calculated as

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}) \quad (3.13a)$$
$$\xi_{n+1}^{trial} = \sigma_{n+1}^{trial} - q_n \quad (3.13b)$$
$$q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}) \quad (3.13c)$$
$$\varepsilon_{n+1}^{p} = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}) \quad (3.13d)$$
$$\alpha_{n+1} = \alpha_n + \Delta\gamma \quad (3.13e)$$
$$f_{n+1} \equiv |\xi_{n+1}| - G(\alpha_{n+1}) = 0 \quad (3.13f)$$
$$\xi_{n+1} = \xi_{n+1}^{trial} - \Delta\gamma sign(\xi_{n+1})(E+H), \quad (3.13g)$$

where the last expression comes about by using $\xi_{n+1} = \sigma_{n+1} - q_{n+1}$.

The set of equations (3.13) are solved in terms of the trial elastic state as follows:

$$|\xi_{n+1}| sign(\xi_{n+1}) = |\xi_{n+1}^{trial}| sign(\xi_{n+1}^{trial}) - \Delta\gamma(E+H) sign(\xi_{n+1}) \quad (3.14)$$

,which is rearranged to give

$$[|\xi_{n+1}| + \Delta\gamma(E+H)] sign(\xi_{n+1}) = |\xi_{n+1}^{trial}| sign(\xi_{n+1}^{trial}). \quad (3.15)$$

Now, since $\Delta\gamma$, E and H are greater than zero, for the above equation to be valid the following two conditions must be true

$$sign(\xi_{n+1}) = sign(\xi_{n+1}^{trial}) \quad (3.16a)$$
$$|\xi_{n+1}| + \Delta\gamma(E+H) = |\xi_{n+1}^{trial}|. \quad (3.16b)$$

It remains to find the consistency parameter $\Delta\gamma > 0$ from the discrete consistency condition (3.13)f. Hence, using (3.16)b and (3.13)f yields

$$\begin{aligned} f_{n+1} &= |\xi_{n+1}^{trial}| - \Delta\gamma(E+H) - G(\alpha_{n+1}) \\ &= |\xi_{n+1}^{trial}| - \Delta\gamma(E+H) - G(\alpha_{n+1}) - G(\alpha_n) + G(\alpha_n) \\ &= |\xi_{n+1}^{trial}| - G(\alpha_n) - \Delta\gamma(E+H) - G(\alpha_{n+1}) + G(\alpha_n) \\ &= f_{n+1}^{trial} - \Delta\gamma(E+H) - G(\alpha_{n+1}) + G(\alpha_n) = 0. \end{aligned} \quad (3.17)$$

The last line of (3.17) is often a nonlinear equation in terms of $\Delta\gamma$ and must be solved by a Newton-Raphson procedure. Once $\Delta\gamma$ is known using (3.16)a in (3.13)abc gives

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial}) \quad (3.18a)$$
$$\xi_{n+1}^{trial} = \sigma_{n+1}^{trial} - q_n \quad (3.18b)$$
$$\xi_{n+1} = \xi_{n+1}^{trial} - \Delta\gamma sign(\xi_{n+1}^{trial})(E+H) \quad (3.18c)$$
$$\varepsilon_{n+1}^{p} = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}^{trial}) \quad (3.18d)$$
$$\alpha_{n+1} = \alpha_n + \Delta\gamma \quad (3.18e)$$
$$q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}^{trial}). \quad (3.18f)$$

Note also that (3.18)c may be rewritten as

$$\xi_{n+1} = \left[1 - \frac{\Delta\gamma(E+H)}{|\xi_{n+1}^{trial}|}\right] \xi_{n+1}^{trial}. \quad (3.19)$$

3.3 The Algorithmic Tangent Modulus

Before summarizing the algorithm, the *consistent* or *algorithmic tangent modulus* is derived. The algorithmic tangent modulus is

$$\mathsf{C}^{(k)}_{n+1} = \frac{\partial \sigma^{(k)}_{n+1}}{\partial \varepsilon^{(k)}_{n+1}}. \tag{3.20}$$

In the following derivation the superscript k is omitted. Keep in mind in the ensuing derivation that ε^p_n, α_n and q_n are constants since they are fixed values determined in the previous step. Some preliminary results are obtained and then subsequently the derivation for the algorithmic tangent modulus is provided. First, differentiate the trial stress σ^{trial}_{n+1}, using $\sigma^{trial}_{n+1} = E(\varepsilon_{n+1} - \varepsilon^p_n)$, to get

$$\frac{\partial \sigma^{trial}_{n+1}}{\partial \varepsilon_{n+1}} = E. \tag{3.21}$$

Second, using $\xi^{trial}_{n+1} = \sigma^{trial}_{n+1} - q_n$ find that

$$\frac{\partial \xi^{trial}_{n+1}}{\partial \varepsilon_{n+1}} = \frac{\partial \sigma^{trial}_{n+1}}{\partial \varepsilon_{n+1}} = E. \tag{3.22}$$

Third, it is necessary to obtain $\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}}$. This is accomplished by using the final result of (3.17), to find

$$\Delta\gamma = \frac{f^{trial}_{n+1} - G(\alpha_{n+1}) + G(\alpha_n)}{E + H}. \tag{3.23}$$

Differentiating (3.23) yields

$$\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} = \frac{1}{E+H}\frac{\partial f^{trial}_{n+1}}{\partial \varepsilon_{n+1}} - \frac{1}{E+H}\frac{\partial G(\alpha_{n+1})}{\partial \varepsilon_{n+1}} + \frac{1}{E+H}\frac{\partial G(\alpha_n)}{\partial \varepsilon_{n+1}} \tag{3.24}$$

From (3.17) notice $f^{trial}_{n+1} \equiv |\xi^{trial}_{n+1}| - G(\alpha_n)$ to obtain

$$\frac{\partial f^{trial}_{n+1}}{\partial \varepsilon_{n+1}} = \frac{\partial |\xi^{trial}_{n+1}|}{\partial \xi^{trial}_{n+1}}\frac{\partial \xi^{trial}_{n+1}}{\partial \varepsilon_{n+1}} = sign(\xi^{trial}_{n+1})E \tag{3.25}$$

Observe also that

$$\frac{\partial G(\alpha_{n+1})}{\partial \varepsilon_{n+1}} = \frac{\partial G}{\partial \alpha_{n+1}}\frac{\partial \alpha_{n+1}}{\partial \Delta\gamma}\frac{\partial \Delta\gamma}{\partial \varepsilon_{n+1}} = \frac{\partial G}{\partial \alpha_{n+1}}(1)\frac{\partial \Delta\gamma}{\partial \varepsilon_{n+1}} = \frac{\partial G}{\partial \alpha_{n+1}}\frac{\partial \Delta\gamma}{\partial \varepsilon_{n+1}} \tag{3.26}$$

and

$$\frac{\partial G(\alpha_n)}{\partial \varepsilon_{n+1}} = 0. \tag{3.27}$$

Now, use (3.25), (3.26) and (3.27) in (3.24) to get

$$\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} + \frac{1}{E+H}\frac{\partial G}{\partial \alpha_{n+1}}\frac{\partial(\Delta\gamma)}{\partial \varepsilon_{n+1}} = \frac{sign(\xi^{trial}_{n+1})E}{E+H}. \tag{3.28}$$

After some algebra (3.28) yields

$$\frac{\partial(\Delta\gamma)}{\partial\varepsilon_{n+1}} = \frac{sign(\xi_{n+1}^{trial})E}{E + \frac{\partial G}{\partial\alpha_{n+1}} + H}. \tag{3.29}$$

Fourth, observe that

$$\begin{aligned}
\sigma_{n+1} &= \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial}) \\
&= (\sigma_{n+1}^{trial} - q_n) + q_n - \Delta\gamma E sign(\xi_{n+1}^{trial}) \\
&= q_n + \xi_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial}) \\
&= q_n + \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] \xi_{n+1}^{trial}.
\end{aligned} \tag{3.30}$$

Finally, differentiate (3.30) with respect to ε_{n+1} and make use of (3.22) and (3.29) as needed to get

$$\begin{aligned}
\frac{\partial\sigma_{n+1}}{\partial\varepsilon_{n+1}} &= \frac{\partial q_n}{\partial\varepsilon_{n+1}} + \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] E + \frac{\partial}{\partial\varepsilon_{n+1}} \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] \xi_{n+1}^{trial} \\
&= \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] E - \frac{\partial\Delta\gamma}{\partial\varepsilon_{n+1}} \frac{E\xi_{n+1}^{trial}}{|\xi_{n+1}^{trial}|} - \Delta\gamma E \frac{\partial|\xi_{n+1}^{trial}|^{-1}}{\partial\varepsilon_{n+1}} \xi_{n+1}^{trial} \\
&= \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] E - \frac{\partial\Delta\gamma}{\partial\varepsilon_{n+1}} \frac{E\xi_{n+1}^{trial}}{|\xi_{n+1}^{trial}|} + \frac{\Delta\gamma E^2}{|\xi_{n+1}^{trial}|^2} sign(\xi_{n+1}^{trial})\xi_{n+1}^{trial} \\
&= \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] E + \frac{\Delta\gamma E^2}{|\xi_{n+1}^{trial}|^2} sign(\xi_{n+1}^{trial})\xi_{n+1}^{trial} - \left(\frac{E^2}{E + \frac{\partial G}{\partial\alpha_{n+1}} + H}\right) \frac{\xi_{n+1}^{trial}}{|\xi_{n+1}^{trial}|} sign(\xi_{n+1}^{trial}) \\
&= \left[1 - \frac{\Delta\gamma E}{|\xi_{n+1}^{trial}|}\right] E + \frac{\Delta\gamma E^2}{|\xi_{n+1}^{trial}|} - \frac{E^2}{E + \frac{\partial G}{\partial\alpha_{n+1}} + H} \\
&= E - \frac{E^2}{E + \frac{\partial G}{\partial\alpha_{n+1}} + H} = \frac{E^2 + E(\frac{\partial G}{\partial\alpha_{n+1}} + H) - E^2}{E + \frac{\partial G}{\partial\alpha_{n+1}} + H} = \frac{E(H + \frac{\partial G}{\partial\alpha_{n+1}})}{E + H + \frac{\partial G}{\partial\alpha_{n+1}}}.
\end{aligned} \tag{3.31}$$

Therefore, in the 1D case, the algorithmic tangent modulus is equivalent to the elasto-plastic tangent modulus (see (3.11)), that is

$$\mathsf{C}_{ep} = \mathsf{C}_{n+1}^{(k)} = \frac{\partial\sigma_{n+1}^{(k)}}{\partial\varepsilon_{n+1}^{(k)}} = \frac{E(H + \frac{\partial G}{\partial\alpha_{n+1}})}{E + H + \frac{\partial G}{\partial\alpha_{n+1}}}. \tag{3.32}$$

In higher dimensions they are not equivalent.

3.4 Summary of results

With all of the above in hand it is possible to summarize the algorithm for 1D plasticity with general isotropic hardening combined with linear kinematic hardening. The algorithm is summarized in Box 3.1.

1. Start with stored known variables $\{\varepsilon_n,\ \varepsilon_n^p,\ \alpha_n,\ q_n\}$.

2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

3. Calculate the elastic trial stress, the trial relative stress, the trial value for the yield function, and check for yielding.

$$
\begin{aligned}
\sigma_{n+1}^{trial} &= E(\varepsilon_{n+1} - \varepsilon_n^p) \\
\varepsilon_{n+1}^{p\ trial} &= \varepsilon_n^p \\
\alpha_{n+1}^{trial} &= \alpha_n \\
\xi_{n+1}^{trial} &= \sigma_{n+1}^{trial} - q_n \\
f_{n+1}^{trial} &= |\xi_{n+1}^{trial}| - G(\alpha_n)
\end{aligned}
$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

set $\sigma_{n+1} = \sigma_{n+1}^{trial}$

set $\mathsf{C}_{ep} = E$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma(E + H) - G(\alpha_{n+1}) + G(\alpha_n) = 0$, solve for $\Delta\gamma$.

$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial})$

$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}^{trial})$

$q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}^{trial})$

$\alpha_{n+1} = \alpha_n + \Delta\gamma$

$\mathsf{C}_{ep} = \frac{E(H+\frac{\partial G}{\partial \alpha})}{E+H+\frac{\partial G}{\partial \alpha}}$

EXIT the algorithm

Box 3.1: 1D Plasticity Algorithm With General Isotropic Hardening Combined With Kinematic Hardening

4 Examples of different types of isotropic hardening combined with *kinematic* hardening

Using the results of section 3.4, different types of isotropic hardening functions, used to create the yield stress function $G(\alpha)$, are combined with kinematic hardening. These cases also demonstrate the actual implementation of the plasticity algorithm of Box 3.1. The yield stress function and its derivative are substituted into the algorithm at the appropriate locations. These algorithms advance the solution variables from their current values at step n to their values at step $n+1$. The resulting algorithm is summarized in a box for each case given. After observing the examples given it is hoped that the reader can then tackle other cases.

4.1 Kinematic Hardening Only (no isotropic hardening)

For this case there is no isotropic hardening. The yield stress function is $G(\alpha) = \sigma_y$ and therefore $\frac{\partial G}{\partial \alpha} = 0$. The algorithm is summarized in Box 4.1. Note that the variable α is not needed for this case. In this case, in step 4 of the algorithm, the consistency parameter, $\Delta\gamma$, is directly solved for algebraically.

4.2 Linear Isotropic Hardening Combined With Kinematic Hardening

The yield stress function is $G(\alpha) = \sigma_y + K\alpha$ and therefore $\frac{\partial G}{\partial \alpha} = K$. The algorithm is summarized in Box 4.2. The consistency parameter, $\Delta\gamma$, is solved for using the last line of (3.17) as follows:

$$\begin{aligned}
&f^{trial}_{n+1} - \Delta\gamma(E+H) - (\sigma_y + K\alpha_{n+1}) + (\sigma_y + K\alpha_n) \\
&= f^{trial}_{n+1} - \Delta\gamma(E+H) - \sigma_y - K\alpha_{n+1} + \sigma_y + K\alpha_n \\
&= f^{trial}_{n+1} - \Delta\gamma(E+H) - K(\alpha_{n+1} - \alpha_n) \\
&= f^{trial}_{n+1} - \Delta\gamma(E+H) - K(\Delta\gamma) = 0.
\end{aligned} \tag{4.1}$$

From the last line of equation (4.1)

$$\Delta\gamma = \frac{f^{trial}_{n+1}}{E+H+K}, \tag{4.2}$$

which is indicated in step 4 of the algorithm given in Box 4.2.

4.3 Exponential Isotropic Hardening Combined With Kinematic Hardening

Voce [7] proposed an exponential form of isotropic hardening. This assumes that the hardening eventually reaches a specified saturation (or maximum) stress. In this case the yield stress function becomes $G(\alpha) = \sigma_y + (\sigma_u - \sigma_y)(1 - e^{-\delta\alpha})$ and therefore $\frac{\partial G}{\partial \alpha} = (\sigma_u - \sigma_y)\delta e^{-\delta\alpha}$. The algorithm is summarized in Box 4.3

1. Start with stored known variables $\{\varepsilon_n,\ \varepsilon_n^p,\ q_n\}$.

2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

3. Calculate the elastic trial stress, the trial relative stress, the trial value for the yield function, and check for yielding.

$$\begin{aligned}
\sigma_{n+1}^{trial} &= E(\varepsilon_{n+1} - \varepsilon_n^p) \\
\varepsilon_{n+1}^{p\ trial} &= \varepsilon_n^p \\
\xi_{n+1}^{trial} &= \sigma_{n+1}^{trial} - q_n \\
f_{n+1}^{trial} &= |\xi_{n+1}^{trial}| - \sigma_y
\end{aligned}$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

set $\sigma_{n+1} = \sigma_{n+1}^{trial}$

set $\mathsf{C}_{ep} = E$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma(E + H) = 0$, solve for $\Delta\gamma$.

$$\Rightarrow \Delta\gamma = \frac{f_{n+1}^{trial}}{E + H}$$

$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial})$

$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}^{trial})$

$q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}^{trial})$

$\mathsf{C}_{ep} = \frac{EH}{E+H}$

EXIT the algorithm

Box 4.1: 1D Plasticity Algorithm With Kinematic Hardening Only

1. Start with stored known variables $\{\varepsilon_n,\ \varepsilon_n^p,\ \alpha_n,\ q_n\}$.

2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

3. Calculate the elastic trial stress, the trial relative stress, the trial value for the yield function, and check for yielding.

$$\begin{aligned}
\sigma_{n+1}^{trial} &= E(\varepsilon_{n+1} - \varepsilon_n^p) \\
\varepsilon_{n+1}^{p\ trial} &= \varepsilon_n^p \\
\alpha_{n+1}^{trial} &= \alpha_n \\
\xi_{n+1}^{trial} &= \sigma_{n+1}^{trial} - q_n \\
f_{n+1}^{trial} &= |\xi_{n+1}^{trial}| - (\sigma_y + K\alpha_n)
\end{aligned}$$

   If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

   set $\sigma_{n+1} = \sigma_{n+1}^{trial}$

   set $\mathsf{C}_{ep} = E$

   EXIT the algorithm

   Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

   continue at step 4

4. Elasto-plastic step

$$\Delta\gamma = \frac{f_{n+1}^{trial}}{E+H+K}$$

$$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial})$$

$$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}^{trial})$$

$$q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}^{trial})$$

$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

$$\mathsf{C}_{ep} = \frac{E(H+K)}{E+H+K}$$

   EXIT the algorithm

Box 4.2: 1D Plasticity Algorithm With Linear Isotropic Hardening Combined With Kinematic Hardening [6]

1. Start with stored known variables $\{\varepsilon_n,\ \varepsilon_n^p,\ \alpha_n,\ q_n\}$.

2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

3. Calculate the elastic trial stress, the trial relative stress, the trial value for the yield function, and check for yielding.

$$
\begin{aligned}
&\sigma_{n+1}^{trial} = E(\varepsilon_{n+1} - \varepsilon_n^p)\\
&\varepsilon_{n+1}^{p\ trial} = \varepsilon_n^p\\
&\alpha_{n+1}^{trial} = \alpha_n\\
&\xi_{n+1}^{trial} = \sigma_{n+1}^{trial} - q_n\\
&C = \sigma_u - \sigma_y\\
&f_{n+1}^{trial} = |\xi_{n+1}^{trial}| - (\sigma_y + C(1 - e^{-\delta\alpha_n}))
\end{aligned}
$$

   If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

   set $\sigma_{n+1} = \sigma_{n+1}^{trial}$

   set $\mathsf{C}_{ep} = E$

   EXIT the algorithm

   Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

   continue at step 4

4. Elasto-plastic step

   Using $f_{n+1}^{trial} - \Delta\gamma(E + H) - G(\alpha_{n+1}) + G(\alpha_n) = 0$, solve for $\Delta\gamma$ by using Newton-Raphson iterations (see Box 4.4).

   $\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial})$

   $\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}^{trial})$

   $q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}^{trial})$

   $\alpha_{n+1} = \alpha_n + \Delta\gamma$

   $\mathsf{C}_{ep} = \frac{E(H + C\delta e^{-\delta\alpha_{n+1}})}{E + H + C\delta e^{-\delta\alpha_{n+1}}}$

   EXIT the algorithm

Box 4.3: 1D Plasticity Algorithm With Exponential Isotropic Hardening Combined With Kinematic Hardening

1. Set $\Delta\gamma = 0$
2. Calculate R, where $R = f_{n+1}^{trial} - \Delta\gamma(E+H) - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$,

   and $G(\alpha) = \sigma_y + C(1 - e^{-\delta\alpha})$
3. Initialize variables

   set $maxiter = 10$

   set $k = 0$, (the iteration counter)

   set $tol = 10^{-5}$

   set $d_g = 0$
4. WHILE $|R| > tol$ and $k < maxiter$

   $\frac{dR}{d\Delta\gamma} = -(E+H) - C\delta e^{-\delta(\alpha_n + \Delta\gamma))}$

   $d_g = -\left[\frac{dR}{d\Delta\gamma}\right]^{-1} R$

   Update $\Delta\gamma = \Delta\gamma + d_g$

   Recalculate $R = f_{n+1}^{trial} - \Delta\gamma(E+H) - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$

   Update $k = k + 1$

   END WHILE
5. END Newton-Raphson iterations

Box 4.4: Newton-Raphson iterations algorithm for exponential isotropic hardening combined with kinematic hardening

4.4 Ramberg-Osgood Isotropic Hardening Combined With Kinematic Hardening

A very common isotropic hardening model for metals is the Ramberg-Osgood [5] model, the form of the equation used here is given by Kojic and Bathe [3]. The yield stress function is $G(\alpha) = \sigma_y + C\alpha^m$ and therefore $\frac{\partial G}{\partial \alpha} = mC\alpha^{m-1}$. The algorithm is summarized in Box 4.5

5 Computer Implementation and Results

A truss program is implemented in MATLAB which includes the possibility of plastic deformations. Only small strains are considered in the implemented program. Several examples are provided. In all cases kinematic hardening is considered. Where included the type of isotopic hardening is indicated.

5.1 Outline of Computer Algorithm – Displacement Control

The following implicit algorithm uses Newton-Raphson iterations within each specified displacement increment to enforce global equilibrium for the truss structure (see Clarke and Hancock [1] and McGuire et al [4] for displacement control details). The specified displacement increments are prescribed at a structure dof chosen by the user. Typically this is the structure dof of maximum displacement in the chosen dof direction. In the algorithm presented, equal size displacement increments are used. The reader is encouraged to note the specific locations where the 1D plasticity routines are introduced into the algorithm. Without the introduction of the plasticity algorithms the computer program would be linear. The algorithm proceeds as follows:

1. Define/initialize variables

   - D_{max} = the user specified maximum displacement at dof q
   - $ninc$ = the user specified number of displacement increments to reach D_{max}
   - $\Delta\bar{u}_q = D_{max}/ninc$ = the specified incremental displacement at dof q
   - $\mathbf{F}$ = the total vector of externally applied global nodal forces
   - $\mathbf{F}^{n+1}$ = the current externally applied global nodal force vector
   - λ^{n+1} = the current load ratio, that is $\lambda^{n+1}\mathbf{F} = \mathbf{F}^n + d\mathbf{F} = \mathbf{F}^n + d\lambda^{n+1}\mathbf{F} = \mathbf{F}^{n+1}$, the load ratio starts out equal to zero
   - $\mathbf{N}$ = the vector of truss axial forces, axial force in truss element i is N_i
   - $\mathbf{u}$ = the vector of global nodal displacements, initially $\mathbf{u} = \mathbf{0}$
   - $\mathbf{x}$ = the vector of nodal x coordinates in the undeformed configuration
   - $\mathbf{y}$ = the vector of nodal y coordinates in the undeformed configuration
   - $\mathbf{L}$ = the vector of truss element lengths based on the current $\mathbf{u}$. L for truss element i is $L_i = \sqrt{((x_2 + u_{x2}) - (x_1 + u_{x1}))^2 + ((y_2 + u_{y2}) - (y_1 + u_{y1}))^2}$. The original element lengths are saved in a vector $\mathbf{L}_o$, where $L_{oi} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The subscripts 1 and 2 refer to node 1 and 2 respectively for a given truss element.

1. Start with stored known variables $\{\varepsilon_n,\ \varepsilon_n^p,\ \alpha_n,\ q_n\}$.

2. An increment of strain gives $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

3. Calculate the elastic trial stress, the trial relative stress, the trial value for the yield function, and check for yielding.

$$\begin{aligned}
\sigma_{n+1}^{trial} &= E(\varepsilon_{n+1} - \varepsilon_n^p) \\
\varepsilon_{n+1}^{p\ trial} &= \varepsilon_n^p \\
\alpha_{n+1}^{trial} &= \alpha_n \\
\xi_{n+1}^{trial} &= \sigma_{n+1}^{trial} - q_n \\
f_{n+1}^{trial} &= |\xi_{n+1}^{trial}| - (\sigma_y + C\alpha_n^m)
\end{aligned}$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

set $\sigma_{n+1} = \sigma_{n+1}^{trial}$

set $\mathsf{C}_{ep} = E$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 4

4. Elasto-plastic step

Using $f_{n+1}^{trial} - \Delta\gamma(E + H) - G(\alpha_{n+1}) + G(\alpha_n) = 0$, solve for $\Delta\gamma$ by using Newton-Raphson iterations (see Box 4.6).

$\sigma_{n+1} = \sigma_{n+1}^{trial} - \Delta\gamma E sign(\xi_{n+1}^{trial})$

$\varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\gamma sign(\xi_{n+1}^{trial})$

$q_{n+1} = q_n + \Delta\gamma H sign(\xi_{n+1}^{trial})$

$\alpha_{n+1} = \alpha_n + \Delta\gamma$

$\mathsf{C}_{ep} = \frac{E(H + mC\alpha_{n+1}^{m-1})}{E + H + mC\alpha_{n+1}^{m-1}}$

EXIT the algorithm

Box 4.5: 1D Plasticity Algorithm With Ramberg-Osgood Isotropic Hardening Combined With Kinematic Hardening

1. Set $\Delta\gamma = 0$
2. Calculate R, where $R = f_{n+1}^{trial} - \Delta\gamma(E+H) - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$,

   and $G(\alpha) = \sigma_y + C\alpha^m$
3. Initialize variables

   set $maxiter = 10$

   set $k = 0$ (the iteration counter)

   set $tol = 10^{-5}$

   set $d_g = 0$
4. WHILE $|R| > tol$ and $k < maxiter$

   $\frac{dR}{d\Delta\gamma} = -(E+H) - mC(\alpha_n + \Delta\gamma)^{m-1}$

   $d_g = -\left[\frac{dR}{d\Delta\gamma}\right]^{-1} R$

   Update $\Delta\gamma = \Delta\gamma + d_g$

   Recalculate $R = f_{n+1}^{trial} - \Delta\gamma(E+H) - G(\alpha_n + \Delta\gamma) + G(\alpha_n)$

   Update $k = k + 1$

   END WHILE
5. END Newton-Raphson iterations

Box 4.6: Newton-Raphson iterations algorithm for Ramberg-Osgood isotropic hardening combined with kinematic hardening

- $\mathbf{c}$ and $\mathbf{s}$ = the vectors of cosines and sines for each truss element angle based on the current $\mathbf{u}$.
- $\mathbf{K} = \mathbf{K_M}$, the assembled global tangent stiffness matrix, where $\mathbf{K_M}$ is the material stiffness which evolves as plastic deformations accumulate in individual truss elements in the truss.
- $\mathbf{K}_s$ = the modified global tangent stiffness matrix to account for supports. Rows and columns associated with zero displacement dofs are set to zero and the diagonal position is set to 1. Other (more efficient) schemes are possible, but this proves simple to implement
- $\boldsymbol{\sigma}^{n+1}$ = the vector of element axial stresses
- $\boldsymbol{q}^{n+1}$ = the vector of element axial back stresses
- $\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_{n+1}^{elast} + \boldsymbol{\varepsilon}_{n+1}^{p}$= the vector of *total* axial strain values for each element i, where $\varepsilon = \frac{1}{L_o}\frac{L^2-L_o^2}{L+L_o}$. Note that this form of calculating strain is better conditioned for numerical calculations and is recommended by Crisfield [2].
- $\boldsymbol{\varepsilon}_{n+1}^{p}$ = the vector of total axial *plastic* strain values for each element i
- C_{ep}^{n+1} = the vector of elasto-plastic moduli for each truss element i
- $\boldsymbol{\alpha}^{n+1}$ = the vector of equivalent plastic strain variables α_i^{n+1} for each truss element i

2. **Start Loop** over load increments (for $n = 0$ to $ninc - 1$).
   (a) Calculate global stiffness matrix $\mathbf{K}$ based on current values of $\mathbf{c}, \mathbf{s}, \mathbf{L}$ and $\mathbf{N}$.
   (b) Modify $\mathbf{K}$ to account for supports and get $\mathbf{K}_s$.
   (c) Calculate the incremental load ratio $d\lambda^{n+1}$. The incremental load ratio is calculated as follows. Calculate a displacement vector based on the current stiffness, that is $\hat{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{F}$. Take from $\hat{\mathbf{u}}$ the displacement in the direction of dof q, that is $\hat{u}_q$. Then $d\lambda^{n+1} = \Delta\bar{u}_q/\hat{u}_q$. Update load ratio $\lambda^{n+1} = \lambda^n + d\lambda^{n+1}$.
   (d) Calculate the incremental force vector $d\mathbf{F} = d\lambda^{n+1}\mathbf{F}$.
   (e) Solve for the incremental global nodal displacements $d\mathbf{u} = \mathbf{K}_s^{-1}d\mathbf{F}$
   (f) Update global nodal displacements, $\mathbf{u}^{n+1} = \mathbf{u}^n + d\mathbf{u}$
   (g) Update $\varepsilon_i = \frac{1}{L_o}\frac{L^2-L_o^2}{L+L_o}$ for each element i and store in $\boldsymbol{\varepsilon}_{n+1}$, note L here is based on $\mathbf{u}^{n+1}$.
   (h) Use chosen plasticity algorithm here to update $\boldsymbol{\varepsilon}_{n+1}^{p}$, $\boldsymbol{\sigma}^{n+1}$, $\boldsymbol{q}^{n+1}$,$\boldsymbol{\alpha}^{n+1}$ and C_{ep}^{n+1}
   (i) Calculate the vector of new internal truss element axial forces $\mathbf{N}^{n+1}$. For truss element i the axial force is $N_i^{n+1} = \sigma_i^{n+1}A_i$.
   (j) Construct the vector of internal global forces $\mathbf{F}_{int}^{n+1}$ based on $\mathbf{N}^{n+1}$.
   (k) Calculate the residual $\mathbf{R} = \lambda^{n+1}\mathbf{F} - \mathbf{F}_{int}^{n+1}$ and modify the residual to account for the required supports.
   (l) Calculate the norm of the residual $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

(m) Iterate for equilibrium if necessary. Set up iteration variables.

- Iteration variable $= k = 0$
- $tolerance = 10^{-6}$
- $maxiter = 100$
- $\delta\mathbf{u} = \mathbf{0}$
- $\delta\lambda = 0$
- Save $\boldsymbol{\varepsilon}^p_{n+1}$, $\boldsymbol{q}^{n+1}$, and $\boldsymbol{\alpha}^{n+1}$ as $\boldsymbol{\varepsilon}^p_{on+1}$, $\boldsymbol{q}^{n+1}_o$, and $\boldsymbol{\alpha}^{n+1}_o$

(n) **Start Iterations** while $R > tolerance$ and $k < maxiter$

i. Calculate the new global stiffness $\mathbf{K}$

ii. Modify the global stiffness to account for supports which gives $\mathbf{K}_s$

iii. Calculate the load ratio correction $\delta\lambda_{k+1}$. The load ratio correction is calculated as follows. Calculate $\breve{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{R}$ and $\hat{\mathbf{u}} = \mathbf{K}_s^{-1}\mathbf{F}$. From $\breve{\mathbf{u}}$ and $\hat{\mathbf{u}}$ extract the component of displacement in the direction of dof q, that is $\breve{u}_q$ and $\hat{u}_q$. Then $\delta\lambda_{k+1} = \delta\lambda_k - \breve{u}_q/\hat{u}_q$.

iv. Calculate the correction to $\mathbf{u}^{n+1}$, which is $\delta\mathbf{u}^{k+1} = \delta\mathbf{u}^k + \mathbf{K}_s^{-1}\left[\mathbf{R} - (\breve{u}_q/\hat{u}_q)\mathbf{F}\right]$, but note that $\mathbf{u}^{n+1}$ is not updated until all iterations are completed

v. Update $\varepsilon_i = \frac{1}{L_o}\frac{L^2-L_o^2}{L+L_o}$ for each element i and store in $\boldsymbol{\varepsilon}_{n+1}$, note L here is based on $\mathbf{u}^{n+1} + \delta\mathbf{u}^{k+1}$.

vi. Reset $\boldsymbol{\varepsilon}^p_{n+1}$, $\boldsymbol{q}^{n+1}$, and $\boldsymbol{\alpha}^{n+1}$ to $\boldsymbol{\varepsilon}^p_{on+1}$, $\boldsymbol{q}^{n+1}_o$, and $\boldsymbol{\alpha}^{n+1}_o$ (see Crisfield [2], pages 154 to 156, for discussion of why this is done).

vii. Use chosen plasticity algorithm here to update $\boldsymbol{\varepsilon}^p_{n+1}$, $\boldsymbol{\sigma}^{n+1}$, $\boldsymbol{q}^{n+1}$, $\boldsymbol{\alpha}^{n+1}$ and C^{n+1}_{ep}.

viii. Calculate the vector of new internal truss element axial forces $\mathbf{N}^{k+1}_{n+1}$. For truss element i the axial force is $(N^{k+1}_{n+1})_i = \sigma^{n+1}_{k+1}A_i$.

ix. Construct the vector of internal global forces $\mathbf{F}^{n+1}_{int}$ based on $\mathbf{N}^{k+1}_{n+1}$.

x. Calculate the residual $\mathbf{R} = (\lambda^{n+1} + \delta\lambda_{k+1})\mathbf{F} - \mathbf{F}^{n+1}_{int}$ and modify the residual to account for the required supports.

xi. $R = \sqrt{\mathbf{R} \bullet \mathbf{R}}$

xii. Update iterations counter $k = k + 1$

(o) **End** of while loop iterations

3. Update variables to their final value for the current increment

- $\lambda^{n+1}_{final} = \lambda^{n+1}_{(0)} + \delta\lambda_k$
- $\mathbf{u}^{n+1}_{final} = \mathbf{u}^{n+1}_{(0)} + \delta\mathbf{u}_{(k)}$

4. **End Loop** over load increments

5.2 Single Truss Element - Monotonic Loading

A single truss element is pin supported at one end and is roller supported at the other end (Figure 1). The resulting truss element is monotonically loaded in tension along its axis

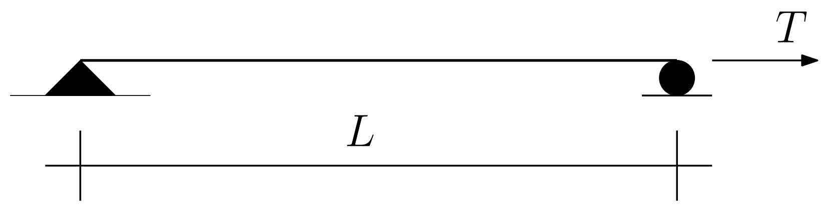

Figure 1: Single truss element loaded in tension.

in the direction of its only free displacement degree of freedom. The example element has $L = 60$ in., $A = 1.0$ in.^{2} and $E = 29 \times 10^3$ ksi. A plot of load versus displacement is provided for three different plasticity models. In all cases the yield stress is taken as $\sigma_y = 36$ ksi. In all three models linear kinematic hardening is included with a modulus of $H = 500$ ksi. For the Ramberg Osgood [3] [5] model, $C = 10.7$ ksi and $m = 0.2$. For the Voce (exponential) [7] model, $\sigma_y = 36$ ksi, $\sigma_u = 58$ ksi, $C = \sigma_u - \sigma_y$ and $\delta = 160$. The nonlinear analysis is achieved by a displacement control procedure. A maximum displacement of 0.5 inches is specified and is achieved by a series of 100 equal displacement increments. The results are presented in Figure 2. The post-yield force increase is evident for the case of kinematic hardening alone. The cases that include isotropic hardening make it difficult to see the contribution of the kinematic hardening since isotropic hardening is happening simultaneously. Monotonic loading also contributes to making it difficult to fully observe the behavior of kinematic hardening. The effect of kinematic hardening and the resulting Bauschinger effect is more evident for an example that includes cyclic loading.

5.3 Single Truss Element - Cyclic Loading

A single truss element is pin supported at one end and is roller supported at the other end (Figure 1). The resulting truss element is cyclically loaded in tension and compression along its axis in the direction of its only free displacement degree of freedom. The example element has $L = 60$ in., $A = 1.0$ in.^{2} and $E = 29 \times 10^3$ ksi. A plot of load versus displacement is provided for a Ramberg Osgood isotropic hardening plasticity model combined with kinematic hardening. The yield stress is taken as $\sigma_y = 36$ ksi. For the Ramberg Osgood model, $C = 30.7$ ksi and $m = 0.2$. The kinematic modulus is $K = 1000$ ksi. The implicit nonlinear analysis with Newton-Raphson iterations for equilibrium at the global level is achieved by a displacement control procedure for the cycles of displacement shown in Figure 3. Notice the highest value of yield force ($P_y \approx 57$ kips) reached at point 1 in Figure 3. Observe that when the bar is loaded to point 2 in the opposite direction the magnitude of force required to reach the yield force ($P_y \approx 42$ kips) is less than at point 1. This difference in *yield* force for tension (+) versus compression (-) is a manifestation of kinematic hardening. The center of the yield surface has shifted from the origin. This behavior would not be observed if only isotropic hardening was employed. The yield force in each direction for points 1 and 2 would have been the same.

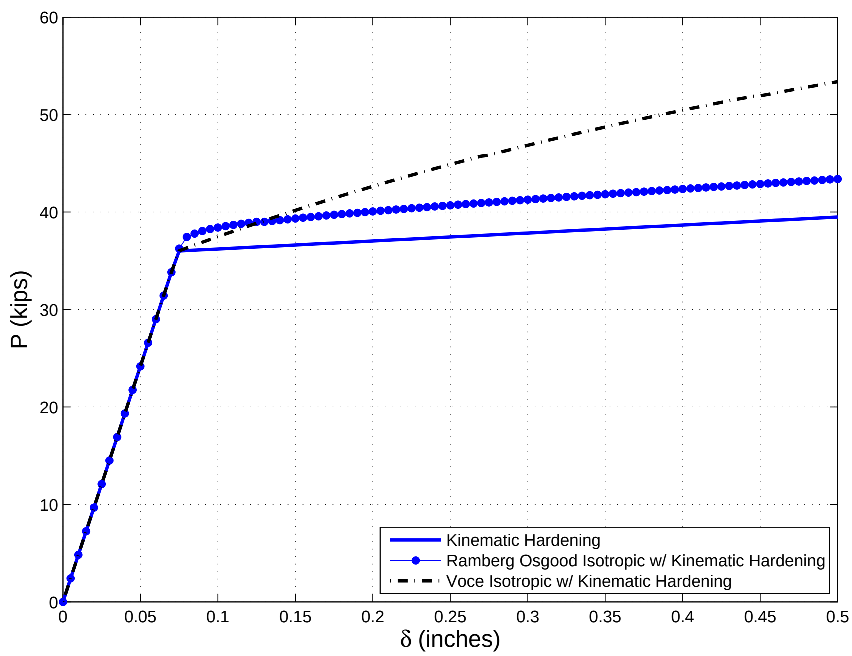

Figure 2: Single truss element monotonically loaded in tension with various 1D plasticity models.

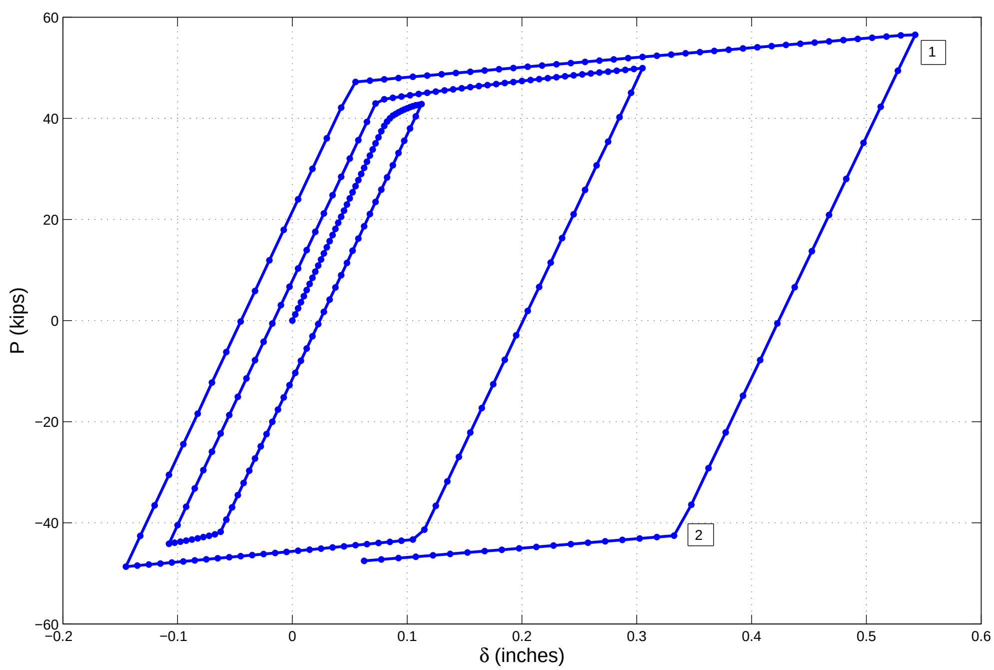

Figure 3: Single truss element cyclically loaded - Ramberg Osgood Isotropic with Kinematic Hardening. Points 1 and 2 have different yield force magnitudes, which is a manifestation of kinematic hardening.

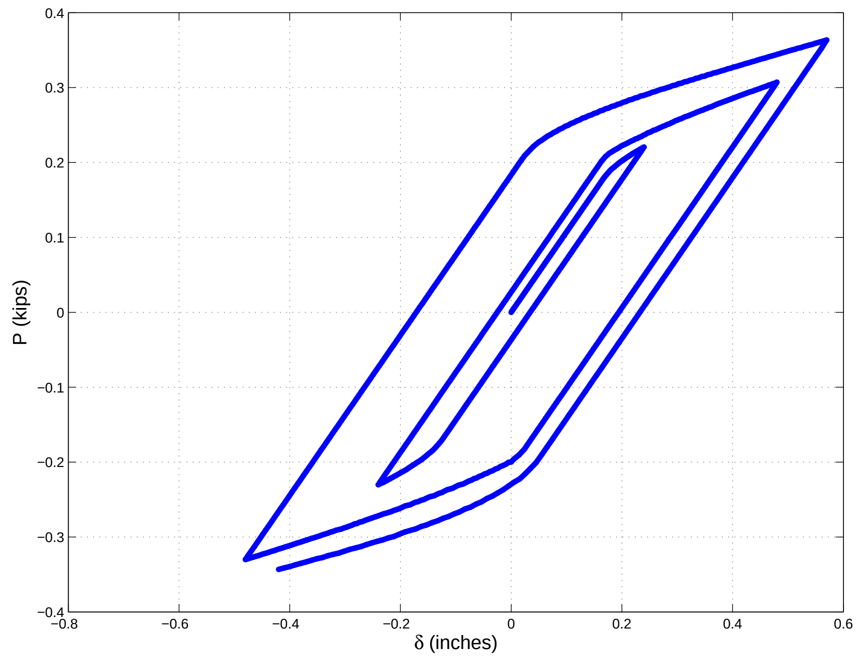

Figure 4: Cantilever Tip Load Versus Free End Vertical Displacement For Cyclically Loaded Truss With Isotropic Voce Material Combined With Kinematic Hardening

5.4 Cantilever Truss - Cylic Loading

A cantilever truss is supported in the x direction at the bottom left node. It is pin supported at the top left node. The truss has 81 members and 42 nodes. The truss is 10 inches long and 0.5 inches tall. Each member has a cross-sectional area of 0.1 in.^{2} and a modulus of elasticity of $E = 29 \times 10^3$ ksi. The truss is loaded(implicit nonlinear analysis) at its right end cyclically by a displacement control scheme. All truss members are modeled with isotropic exponential hardening by Voce [7] combined with kinematic hardening. The material model values are $\sigma_y = 36$ ksi, $\sigma_u = 58$ ksi, $C = \sigma_u - \sigma_y$, $\delta = 160$, and $H = 5000$ ksi. A plot of load versus displacement is provided in Figure 4. The truss model is shown with its deflected shape in Figure 5.

6 Conclusions

Plasticity models are often used in finite element analysis programs in order to account for nonlinear material behavior. In truss programs 1D plasticity is necessary to model element behavior when plastic material behavior is possible. The present paper extends the material behavior described in [8] in order to include kinematic hardening. The current paper provides numerous derivations, material algorithms, and examples of 1D isotropic hardening plasticity combined with kinematic hardening. An algorithm is provided which describes how such models are incorporated into a finite element truss program. Numerical results are provided for a single bar element undergoing monotonic loading and also a case for cyclic

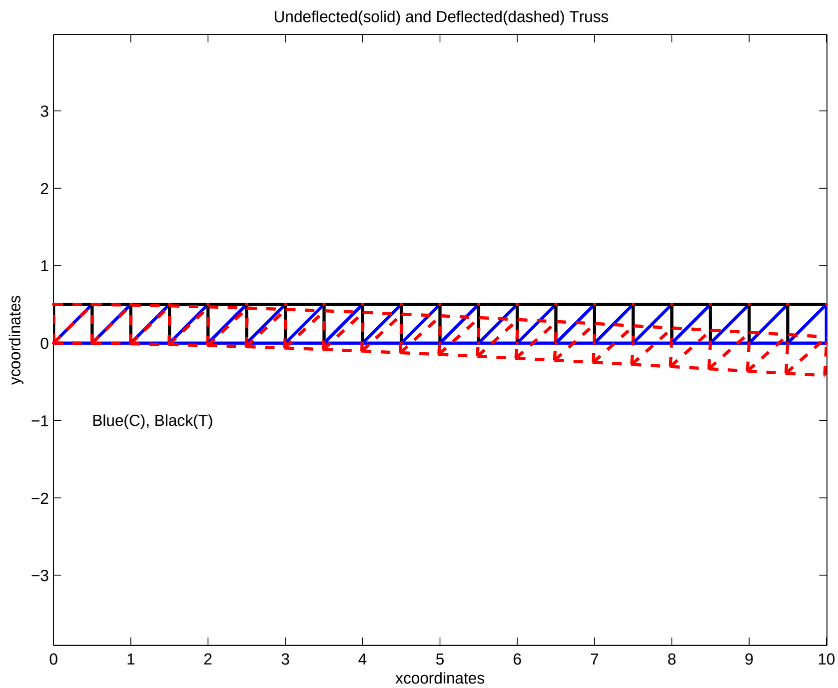

Figure 5: Original and Deflected Shape of Cantilever Truss

loading. Numerical results are also provided for the case of a cyclically loaded cantilevered truss. This introductory paper provides information as a stepping stone toward 3D plasticity models.

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