3D Computational J2 Plasticity Algorithm With Hardening

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1 Introduction

Summary of the algorithm for 3D plasticity is presented. The algorithm includes the possibility of nonlinear isotropic and kinematic hardening in conjunction with J2 plasticity. An algorithm of this type is most appropriate for ductile metals. Much of the information presented is found in the book, Computational Inelasticity, by Simo and Hughes [7].

1.1 Isotropic Linear Elasticity

For the isotropic linear elastic portion of the material behavior, the relation between stress and strain is

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}, \tag{1}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{2}$$

where the 4th order isotropic elasticity tensor is written as

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{3}$$

Substituting (3) into (2) and simplifying yields

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + \mu(\varepsilon_{ij} + \varepsilon_{ji}). \tag{4}$$

The relationship between stress and strain is expressed in terms of Lamé parameters in equation (4). An alternative representation is to express stress in terms of volumetric and deviatoric components. That is

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{vol} + \boldsymbol{\sigma}_{dev} = \kappa\mathbf{1}\otimes\mathbf{1} : \boldsymbol{\varepsilon} + 2\mu\mathbf{I}_{dev} : \boldsymbol{\varepsilon}, \tag{5}$$

where κ is the bulk modulus.

1.2 Isotropic and Kinematic Hardening Equations

A fairly general expression, which includes a nonlinear term [10], for isotropic hardening takes the following form:

$$K(\alpha) = \sigma_y + \theta \bar{H}\alpha + (\sigma_u - \sigma_y)(1 - e^{-\delta\alpha}), \tag{6}$$

where $\theta \in [0, 1]$ and is used to facility mixed hardening. The hardening modulus is $\bar{H}$. Depending on the choice of θ kinematic hardening may also be included. Although a nonlinear form could be used, a linear form of kinematic hardening is

$$H(\alpha) = (1 - \theta)\bar{H}\alpha. \tag{7}$$

In each of the above cases the following terms are found:

$$\frac{dK(\alpha)}{d\alpha} = K' = \theta\bar{H} + (\sigma_u - \sigma_y)\delta e^{-\delta\alpha} \tag{8}$$

and

$$\frac{dH(\alpha)}{d\alpha} = H' = (1 - \theta)\bar{H}. \tag{9}$$

These terms are included in the plasticity algorithm of the following section.

1.3 Algorithm for 3D J2 plasticity with nonlinear forms of hardening [7]

A 3D J2 plasticity algorithm with nonlinear isotropic and kinematic hardening is contained in Box 1.1. It is important to note that the algorithms are expressed in terms of tensor notation.

Box 1.1: 3D Plasticity Algorithm With General Isotropic Hardening (Tensor Notation)

1. Start with stored known variables $\{\boldsymbol{\varepsilon}_n,\ \boldsymbol{e}^p_n,\ \alpha_n,\ \boldsymbol{\beta}_n\}$.
2. An increment of strain gives $\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_n + \Delta\boldsymbol{\varepsilon}_n$.
3. Calculate the trial elastic stress.

$$dev[\boldsymbol{\varepsilon}]_{n+1} = \boldsymbol{e}_{n+1} = \boldsymbol{\varepsilon}_{n+1} - \frac{1}{3}(tr[\boldsymbol{\varepsilon}_{n+1}])\mathbf{1}$$

$$\boldsymbol{s}^{trial}_{n+1} = 2\mu(\boldsymbol{e}_{n+1} - \boldsymbol{e}^p_n)$$

$$\boldsymbol{\xi}^{trial}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - \boldsymbol{\beta}_n$$

4. Check yield condition

$$f^{trial}_{n+1} = \left\|\boldsymbol{\xi}^{trial}_{n+1}\right\| - \sqrt{\frac{2}{3}}K(\alpha_n)$$

If $f_{n+1}^{trial} \leq 0$ then the load step is elastic

set $\boldsymbol{\sigma}_{n+1} = \kappa tr[\boldsymbol{\varepsilon}_{n+1}]\mathbf{1} + \boldsymbol{s}_{n+1}^{trial}$

set $\mathbf{C}_{n+1} = \mathbf{C} = \kappa \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}_{dev}$

EXIT the algorithm

Else $f_{n+1}^{trial} > 0$ and hence the load step is elasto-plastic

continue at step 5

5. Elasto-plastic step

$\mathbf{n}_{n+1} = \frac{\boldsymbol{\xi}_{n+1}^{trial}}{\|\boldsymbol{\xi}_{n+1}^{trial}\|}$

Find $\Delta\gamma$ from consistency condition, Box 1.2.

Update $\alpha_{n+1} = \alpha_n + \sqrt{\frac{2}{3}}\Delta\gamma$

6. Update back stress, plastic strain, and stress

$\boldsymbol{\beta}_{n+1} = \boldsymbol{\beta}_n + \sqrt{\frac{2}{3}}[H(\alpha_{n+1}) - H(\alpha_n)]\mathbf{n}_{n+1}$

$\boldsymbol{e}_{n+1}^p = \boldsymbol{e}_n^p + \Delta\gamma \mathbf{n}_{n+1}$

$\boldsymbol{\sigma}_{n+1} = \kappa tr[\boldsymbol{\varepsilon}_{n+1}]\mathbf{1} + \boldsymbol{s}_{n+1}^{trial} - 2\mu\Delta\gamma\mathbf{n}_{n+1}$

7. Compute the consistent elasto-plastic tangent moduli

$\theta_{n+1} = 1 - \frac{2\mu\Delta\gamma}{\|\boldsymbol{\xi}_{n+1}^{trial}\|}$

$\bar{\theta}_{n+1} = \frac{1}{1+\frac{[K'+H']_{n+1}}{3\mu}} - (1 - \theta_{n+1})$

set $\mathbf{C}_{n+1} = \kappa \mathbf{1} \otimes \mathbf{1} + 2\mu\theta_{n+1}[\mathbf{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}] - 2\mu\bar{\theta}_{n+1}\mathbf{n}_{n+1} \otimes \mathbf{n}_{n+1}$

1.4 Newton-Raphson iterations for consistency [7]

The consistency condition is satisfied when the correct value of the algorithmic consistency parameter, $\Delta\gamma$, is found. The consistency condition for 3D J2 plasticity with nonlinear hardening is

$$g(\Delta\gamma) = -\sqrt{\frac{2}{3}}K(\alpha_{n+1}) + \|\boldsymbol{\xi}_{n+1}^{trial}\| - (2\mu\Delta\gamma + \sqrt{\frac{2}{3}}[H(\alpha_{n+1}) - H(\alpha_n)]) = 0. \tag{10}$$

Box 1.2: Newton-Raphson iterations for consistency (tensor notation)

1. Initialize variables

$\Delta\gamma^{(0)} = 0$

$\alpha_{n+1}^{(0)} = \alpha_n$

$k = 0$

$maxiter = 20$

$tol = 10^{-4}$

set $g = 1$ (arbitrary value, to force consistency check and iterations)

2. Iterate, while $|g(\Delta\gamma^{(k)})| > tol$ and $k \leq maxiter$

   $k = k + 1$

   (i) Compute $\Delta\gamma^{(k)}$

   $g(\Delta\gamma^{(k)}) = -\sqrt{\frac{2}{3}}K(\alpha_{n+1}^{(k)}) + \left\|\boldsymbol{\xi}_{n+1}^{trial}\right\| - (2\mu\Delta\gamma^{(k)} + \sqrt{\frac{2}{3}}[H(\alpha_{n+1}^{(k)}) - H(\alpha_n)])$

   $Dg \equiv \frac{dg(\Delta\gamma^{(k)})}{d\Delta\gamma} = -2\mu\left\{1 + \frac{H'(\alpha_{n+1}^{(k)}) + K'(\alpha_{n+1}^{(k)})}{3\mu}\right\}$

   $\Delta\gamma^{(k+1)} = \Delta\gamma^{(k)} - \frac{g(\Delta\gamma^{(k)})}{Dg}$

   (ii) Update equivalent plastic strain

   $$\alpha_{n+1}^{(k+1)} = \alpha_n + \sqrt{\frac{2}{3}}\Delta\gamma^{(k+1)}$$

   End

1.5 Remarks

1. Elasticity constants are summarized here:

   (a) Bulk modulus, κ

   $$\kappa = \lambda + \frac{2}{3}\mu \tag{11}$$

   (b) Shear modulus, $\mu = G$

   $$\mu = G = \frac{E}{2(1+\nu)} \tag{12}$$

   (c) Lamé parameter, λ

   $$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{13}$$

   (d) Modulus of elasticity, E

   (e) Poisson's ratio, ν

2. Identity tensors used in Box 1.1 are summarized here:

   (a) Second order symmetric unit tensor, $\mathbf{1}$

   $$\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \tag{14}$$

(b) Fourth order symmetric unit tensor, $\mathbf{I}$. For arbitrary 2nd order tensor $\mathbf{S}$, $\mathbf{I} : \mathbf{S} = \mathbf{S}^{sym}$. That is, the operation results in the symmetric part of the tensor.

$$\mathbf{I} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \tag{15}$$

(c) Fourth order identity tensor, $\mathbf{1} \otimes \mathbf{1}$. For arbitrary 2nd order tensor $\mathbf{S}$, $\mathbf{1} \otimes \mathbf{1} : \mathbf{S} = tr(S)\mathbf{1}$.

$$\mathbf{1} \otimes \mathbf{1} = \delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \tag{16}$$

(d) Fourth order deviatoric identity tensor. For arbitrary 2nd order tensor $\mathbf{S}$, $\mathbf{I}_{dev} : \mathbf{S} = \mathbf{S}^{dev}$. That is, the operation results in the deviatoric part of the tensor.

$$\mathbf{I}_{dev} = \mathbf{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1} \tag{17}$$

3. Boxes 1.1 and 1.2 are written making use of tensor notation.

4. In Box 1.1 the term, $[K' + H']_{n+1}$,in the algorithmic tangent modulus, is indicating that K' and H' be evaluated at α_{n+1}.

5. The tensor $\mathbf{C}_{n+1}$ is the consistent(or algorithmic) elato-plastic tangent modulus. It is the modulus to be used when constructing the contribution to the element stiffness matrix at a particular gauss point of integration. By using this modulus, in a nonlinear analysis, the quadratic convergence of Newton-Raphson iterations at the global level is maintained, for appropriate step sizes.

2 Algorithms using Voigt notation

It is common to implement finite element computer programs by using Voigt notation [1]. Hence, it is convenient to express the previous algorithms in the Voigt form. With an eye toward that eventual goal, some variables in Voigt notation are presented below.

2.1 2nd order tensors to Voigt form

For stress, a 2nd order tensor to Voigt form is as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Rightarrow \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}. \tag{18}$$

For tensorial strains, the Voigt form is as follows:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}. \tag{19}$$

The Voigt form, in terms of engineering shear strains, is as follows:

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}. \tag{20}$$

2.2 4th order tensors to Voigt form

$$\mathbf{1} \otimes \mathbf{1} \Rightarrow \mathbf{1}\mathbf{1}^T = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

$$\mathbf{I}_{dev} \Rightarrow \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{22}$$

$$\mathbf{I} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{23}$$

$$\mathbf{I}_2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{24}$$

2.3 A needed transformation matrix [3]

$$\mathbf{P} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{25}$$

3 Matrix Voigt representation of algorithms for implementation with FEA

A usual FEA implementation is done using matrix Voigt notation and engineering shear strains rather than tensor shear strains. Furthermore, it is convenient to construct the algorithms to accept engineering shear strains as input and output a consistent algorithmic tangent modulus for use within an element stiffness formulated with engineering strains. Algorithms which meet these requirements are provided in Boxes 3.1 and 3.2 below.

3.1 FEA Matrix Voigt Algorithm for 3D J2 plasticity with nonlinear forms of hardening [7][3]

A 3D J2 plasticity algorithm with nonlinear isotropic and kinematic hardening is contained in Box 3.1. In this context stresses and strains are 6 by 1 column vectors and 4th order identity tensors and moduli matrices are 6 by 6 matrices. Examples of these entities are given in equations (18) through (25).

Box 3.1: 3D Plasticity Algorithm With General Isotropic Hardening (Matrix Voigt Notation)

1. Start with stored known variables $\{\boldsymbol{\epsilon}_n,\ \boldsymbol{\epsilon}^p_n,\ \alpha_n,\ \boldsymbol{\beta}_n\}$. Convert engineering strains to tensorial strains.

$$\boldsymbol{\varepsilon}_n = \mathbf{I}\boldsymbol{\epsilon}_n,\ \boldsymbol{e}^p_n = \mathbf{I}\boldsymbol{\epsilon}^p_n,\ \Delta\boldsymbol{\varepsilon}_n = \mathbf{I}\Delta\boldsymbol{\epsilon}_n$$

2. An increment of strain gives $\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_n + \Delta\boldsymbol{\varepsilon}_n$.
3. Calculate the trial elastic stress.

$$dev[\boldsymbol{\varepsilon}]_{n+1} = \boldsymbol{e}_{n+1} = \boldsymbol{\varepsilon}_{n+1} - \frac{1}{3}(tr[\boldsymbol{\varepsilon}_{n+1}])\mathbf{1}$$

$$\boldsymbol{s}^{trial}_{n+1} = 2\mu(\boldsymbol{e}_{n+1} - \boldsymbol{e}^p_n)$$

$$\boldsymbol{\xi}^{trial}_{n+1} = \boldsymbol{s}^{trial}_{n+1} - \boldsymbol{\beta}_n$$

4. Check yield condition

$$f^{trial}_{n+1} = \sqrt{(\boldsymbol{\xi}^{trial}_{n+1})^T\mathbf{P}\boldsymbol{\xi}^{trial}_{n+1}} - \sqrt{\frac{2}{3}}K(\alpha_n)$$

   If $f^{trial}_{n+1} \leq 0$ then the load step is elastic

   set $\boldsymbol{\sigma}_{n+1} = \kappa tr[\boldsymbol{\varepsilon}_{n+1}]\mathbf{1} + \boldsymbol{s}^{trial}_{n+1}$

   set $\mathbf{C}_{n+1} = \mathbf{C} = \kappa\mathbf{1}\mathbf{1}^T + 2\mu\mathbf{I}_{dev}$

   EXIT the algorithm

   Else $f^{trial}_{n+1} > 0$ and hence the load step is elasto-plastic

   continue at step 5

5. Elasto-plastic step

   $\mathbf{n}_{n+1} = \frac{\boldsymbol{\xi}^{trial}_{n+1}}{\sqrt{(\boldsymbol{\xi}^{trial}_{n+1})^T\mathbf{P}\boldsymbol{\xi}^{trial}_{n+1}}}$

   Find $\Delta\gamma$ from consistency condition, Box 3.2.

   Update $\alpha_{n+1} = \alpha_n + \sqrt{\frac{2}{3}}\Delta\gamma$

6. Update back stress, plastic strain, and stress

   $\boldsymbol{\beta}_{n+1} = \boldsymbol{\beta}_n + \sqrt{\frac{2}{3}}[H(\alpha_{n+1}) - H(\alpha_n)]\mathbf{n}_{n+1}$

   $\boldsymbol{e}^p_{n+1} = \boldsymbol{e}^p_n + \Delta\gamma\mathbf{n}_{n+1}$

   $\boldsymbol{\sigma}_{n+1} = \kappa tr[\boldsymbol{\varepsilon}_{n+1}]\mathbf{1} + \boldsymbol{s}^{trial}_{n+1} - 2\mu\Delta\gamma\mathbf{n}_{n+1}$

7. Compute the consistent elasto-plastic tangent moduli

   $\theta_{n+1} = 1 - \frac{2\mu\Delta\gamma}{\sqrt{(\boldsymbol{\xi}^{trial}_{n+1})^T\mathbf{P}\boldsymbol{\xi}^{trial}_{n+1}}}$

   $\bar{\theta}_{n+1} = \frac{1}{1+\frac{[K'+H']_{n+1}}{3\mu}} - (1 - \theta_{n+1})$

   $\mathbf{C}_{n+1} = \kappa\mathbf{1}\mathbf{1}^T + 2\mu\theta_{n+1}\mathbf{I}_{dev} - 2\mu\bar{\theta}_{n+1}\mathbf{n}_{n+1}\mathbf{n}^T_{n+1}$

   EXIT algorithm

8. Convert terms so output is in form consistent with engineering shear strains

$$\text{set } \boldsymbol{\epsilon}_{\text{n+1}} = \mathbf{I}_2 \boldsymbol{\varepsilon}_{\text{n+1}},\ \boldsymbol{\epsilon}^{\text{p}}_{\text{n+1}} = \mathbf{I}_2 \boldsymbol{e}^{\text{p}}_{\text{n+1}}$$

3.2 FEA Newton-Raphson iterations for consistency [7][3]

The consistency condition is satisfied when the correct value of the algorithmic consistency parameter, $\Delta\gamma$, is found. The consistency condition for 3D J2 plasticity with nonlinear hardening is

$$g(\Delta\gamma) = -\sqrt{\frac{2}{3}}K(\alpha_{n+1}) + \sqrt{(\boldsymbol{\xi}^{trial}_{n+1})^T \mathbf{P} \boldsymbol{\xi}^{trial}_{n+1}} - (2\mu\Delta\gamma + \sqrt{\frac{2}{3}}[H(\alpha_{n+1}) - H(\alpha_n)]) = 0. \quad (26)$$

Box 3.2: Newton-Raphson iterations for consistency (Matrix Voigt notation)

1. Initialize variables

$$\Delta\gamma^{(0)} = 0$$
$$\alpha^{(0)}_{n+1} = \alpha_n$$
$$k = 0$$
$$maxiter = 20$$
$$tol = 10^{-4}$$

set $g = 1$ (arbitrary value, to force consistency check and iterations)

2. Iterate, while $|g(\Delta\gamma^{(k)})| > tol$ and $k \leq maxiter$

$$k = k + 1$$

(i) Compute $\Delta\gamma^{(k)}$

$$g(\Delta\gamma^{(k)}) = -\sqrt{\frac{2}{3}}K(\alpha^{(k)}_{n+1}) + \sqrt{(\boldsymbol{\xi}^{trial}_{n+1})^T \mathbf{P} \boldsymbol{\xi}^{trial}_{n+1}} - (2\mu\Delta\gamma^{(k)} + \sqrt{\frac{2}{3}}[H(\alpha^{(k)}_{n+1}) - H(\alpha_n)])$$

$$Dg \equiv \frac{dg(\Delta\gamma^{(k)})}{d\Delta\gamma} = -2\mu\left\{1 + \frac{H'(\alpha^{(k)}_{n+1}) + K'(\alpha^{(k)}_{n+1})}{3\mu}\right\}$$

$$\Delta\gamma^{(k+1)} = \Delta\gamma^{(k)} - \frac{g(\Delta\gamma^{(k)})}{Dg}$$

(ii) Update equivalent plastic strain

$$\alpha^{(k+1)}_{n+1} = \alpha_n + \sqrt{\tfrac{2}{3}}\Delta\gamma^{(k+1)}$$

End

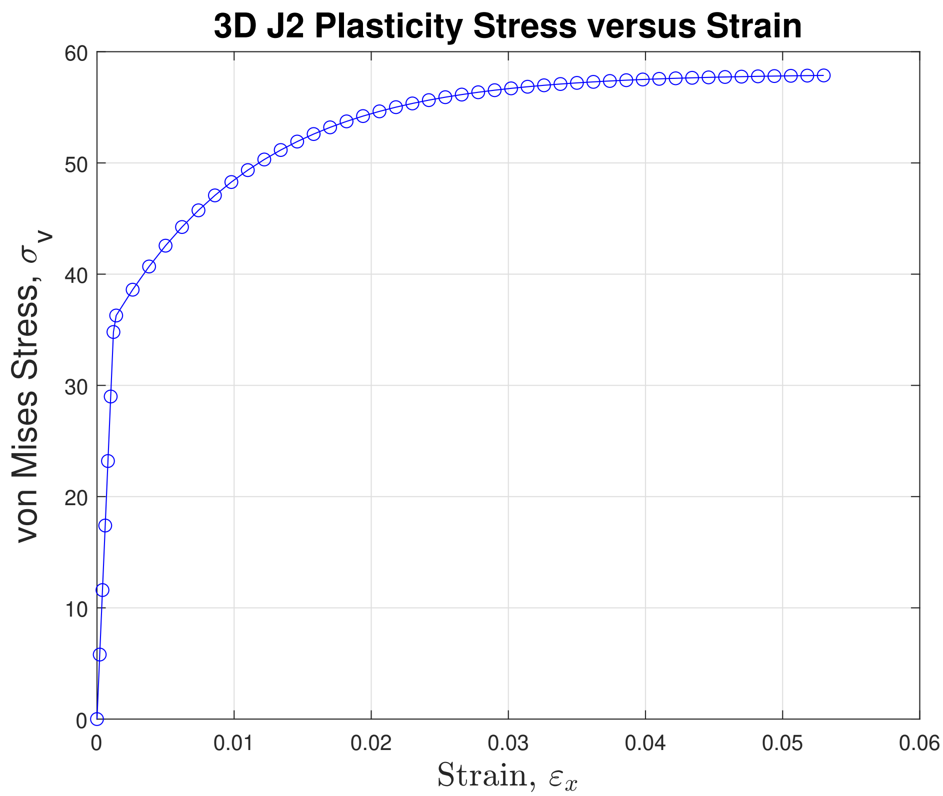

Figure 1: von Mises stress versus axial strain for material point in tension

4 Numerical Examples

4.1 Tension at a material point with 3D J2 plasticity

A material point has the following properties. Modulus $E = 29000$ ksi, elastic $\nu = 0.3$, in plastic range $\nu_p = 0.5$. For hardening, due to Voce [10], $\delta = 100$, $\sigma_y = 36$ ksi, and $\sigma_u = 58$ ksi. For this example the hardening modulus $\bar{H} = 0$. Notice the values of ν change in the plastic range. This occurs naturally in the algorithm, but the incremental strains must be specified properly for a material point to get expected behavior like a 1D bar in tension. In the elastic region (von Mises stress $\leq \sigma_y$), the nonzero incremental strains are $\epsilon_{11} = 0.002$, $\epsilon_{22} = -0.002\nu$, $\epsilon_{33} = -0.002\nu$. However, when the von Mises stress exceeds σ_y, the incremental strains are $\epsilon_{11} = 0.002$, $\epsilon_{22} = -0.002\nu_p$, $\epsilon_{33} = -0.002\nu_p$. In this example, 50 increments of strain are applied to the material point. This is done within a short Matlab script which checks when the von Mises stress exceeds yield and adjusts ν for the increments accordingly. The results are illustrated in Figure 1. Since this case only includes axial strains the results are identical regardless of using tensorial strains or engineering strains, since the engineering shear strains do not participate in the results. As expected the material yields at $\sigma_y = 36$ ksi and hardening continues until reaching an ultimate stress of $\sigma_u = 58$ ksi.

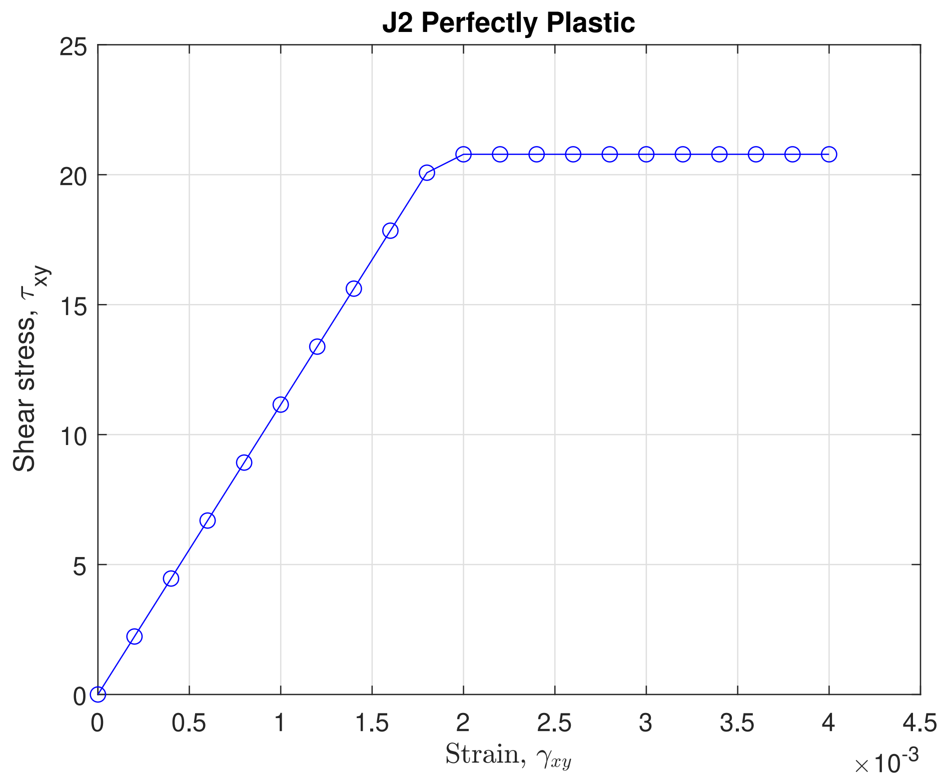

Figure 2: Perfectly plastic shear stress versus engineering shear strain

4.2 Shear at a material point with 3D J2 plasticity

A material point has the following properties. Modulus $E = 29000$ ksi, elastic $\nu = 0.3$. For the case of perfect plasticity there is no hardening, hence, setting $\sigma_y = \sigma_u = 36$ ksi eliminates Voce hardening. Furthermore, the linear hardening modulus is also set to zero, $\bar{H} = 0$. For this case, the nonzero incremental strain is $\gamma_{12} = 0.002$. In this example, 20 increments of strain are applied to the material point. This is done within a short Matlab script. The results are illustrated in Figure 2. It is an interesting verification case for which it is observed that $\tau = G\gamma$ for the linear elastic portion of the plot. It is also noteworthy that shear yielding, τ_y occurs at a value of $\frac{1}{\sqrt{3}}\sigma_y$, which is in line with the prediction of von Mises (J2) plasticity theory. These observations provide verification that the algorithm is performing correctly within an engineering strain formulation.

5 Conclusion

Computational J2 plasticity with mixed nonlinear hardening is presented. Algorithms to implement J2 plasticity into the finite element method or virtual element method, for example, are given. Numerical results and relevant references are provided.

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