

# Covariant Differentiation

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## 1 Introduction

To set the context for how to do covariant differentiation we first need to lay some ground work. The notation closely follows that of Green and Zerna [1]. The Einstein summation convention is assumed throughout. Rather than deriving why covariant differentiation works and why it matters, instead we present the process of doing covariant differentiation and also give some applications.

## 2 Noteworthy Observations

A variety of observations are used in the content provided herein. These observations are noted here:

- (a) a vertical bar indicates covariant differentiation with respect to the subscript that follows the vertical bar
- (b) The covariant derivative of any metric tensor is zero,  $g^{ij}|_d = g_{ij|d} = 0$ . This is sometimes referred to as metric compatibility. As a result, such terms can be eliminated, or sometimes it is useful to write  $g_{ij|d} = g_{ij,d} + \Gamma_{id}^m g_{mj} + \Gamma_{jd}^m g_{im} = 0$  and use it by rearrangement to replace terms like  $g_{ij,d}$  that may arise in derivations.
- (c) second order Christoffel symbols are symmetric with respect to the bottom indices,  $\Gamma_{ij}^k = \Gamma_{ji}^k$
- (d) since Christoffel symbols are not tensors, covariant derivatives act like normal derivatives when they act on Christoffel symbols. That is,  $\Gamma_{ij|r}^k = \Gamma_{ij,r}^k$

## 3 Cylindrical Coordinates

Cartesian coordinates in terms of cylindrical coordinates are (see Figure 1)

$$\begin{aligned}x^1 &= \theta^1 \cos \theta^2 \\x^2 &= \theta^1 \sin \theta^2 \\x^3 &= \theta^3\end{aligned}\tag{1}$$

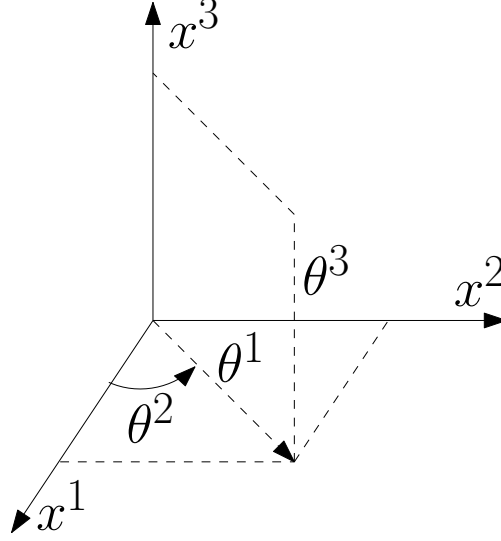


Figure 1: Cylindrical Coordinates

Cylindrical coordinates in terms of Cartesian coordinates are

$$\begin{aligned}\theta^1 &= \sqrt{(x^1)^2 + (x^2)^2} \\ \theta^2 &= \tan^{-1} \left( \frac{x^2}{x^1} \right) \\ \theta^3 &= x^3\end{aligned}\tag{2}$$

### 3.1 Covariant Base Vectors

Covariant base vectors are tangent to the curvilinear coordinate curves,  $\theta^i$ . The covariant base vectors are found according to the following formula

$$\mathbf{g}_j = \frac{\partial x^r}{\partial \theta^j} \mathbf{i}_r,\tag{3}$$

where  $\mathbf{i}_r = \mathbf{i}^r$  are the unit Cartesian base vectors and  $\mathbf{i}_r \cdot \mathbf{i}^s = \delta_s^r$ . Consequently, each of the covariant base vectors are

$$\begin{aligned}\mathbf{g}_1 &= \frac{\partial x^1}{\partial \theta^1} \mathbf{i}_1 + \frac{\partial x^2}{\partial \theta^1} \mathbf{i}_2 + \frac{\partial x^3}{\partial \theta^1} \mathbf{i}_3 = \cos \theta^2 \mathbf{i}_1 + \sin \theta^2 \mathbf{i}_2 + 0 \mathbf{i}_3 \\ \mathbf{g}_2 &= \frac{\partial x^r}{\partial \theta^2} \mathbf{i}_r = -\theta^1 \sin \theta^2 \mathbf{i}_1 + \theta^1 \cos \theta^2 \mathbf{i}_2 + 0 \mathbf{i}_3 \\ \mathbf{g}_3 &= \frac{\partial x^r}{\partial \theta^3} \mathbf{i}_r = 0 \mathbf{i}_1 + 0 \mathbf{i}_2 + 1 \mathbf{i}_3\end{aligned}\tag{4}$$

### 3.2 Contravariant Base Vectors

Contravariant base vectors are perpendicular to the coordinate surfaces,  $\theta^i$ , at a point. The contravariant base vectors are found according to the following formula

$$\mathbf{g}^j = \frac{\partial \theta^j}{\partial x^r} \mathbf{i}^r. \quad (5)$$

Consequently, each of the covariant base vectors are

$$\begin{aligned} \mathbf{g}^1 &= \frac{\partial \theta^1}{\partial x^r} \mathbf{i}^r = \cos \theta^2 \mathbf{i}^1 + \sin \theta^2 \mathbf{i}^2 + 0 \mathbf{i}^3 \\ \mathbf{g}^2 &= \frac{\partial \theta^2}{\partial x^r} \mathbf{i}^r = -\frac{\sin \theta^2}{\theta^1} \mathbf{i}^1 + \frac{\cos \theta^2}{\theta^1} \mathbf{i}^2 + 0 \mathbf{i}^3 \\ \mathbf{g}^3 &= \frac{\partial \theta^3}{\partial x^r} \mathbf{i}^r = 0 \mathbf{i}^1 + 0 \mathbf{i}^2 + 1 \mathbf{i}^3 \end{aligned} \quad (6)$$

### 3.3 The Metric Tensors

The covariant metric tensor has components

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = g_{ji}. \quad (7)$$

In matrix form

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\theta^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (8)$$

The contravariant metric tensor has components

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = g_{ij}^{-1} = g^{ji}. \quad (9)$$

In matrix form

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{(\theta^1)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

The mixed metric tensor has components

$$g_i^j = \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j. \quad (11)$$

In matrix form

$$[g_i^j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (12)$$

### 3.4 Distance Using the Metric Tensor

The differential distance  $ds$  is found using the metric tensor as follows:

$$ds^2 = g_{ij} dx^i dx^j. \quad (13)$$

### 3.5 Christoffel Symbols of the Second Kind

The Christoffel symbols of the second kind are found according to the formula

$$\Gamma_{ij}^r = \frac{1}{2} g^{rs} (g_{is,j} + g_{js,i} - g_{ij,s}) = \mathbf{g}^r \cdot \mathbf{g}_{i,j}. \quad (14)$$

For cylindrical coordinates the Christoffel symbols are found to be

$$\begin{aligned} \Gamma_{11}^1 &= 0, \quad \Gamma_{11}^2 = 0, \quad \Gamma_{11}^3 = 0, \\ \Gamma_{21}^1 &= 0, \quad \Gamma_{21}^2 = \frac{1}{\theta^1}, \quad \Gamma_{21}^3 = 0, \\ \Gamma_{31}^1 &= 0, \quad \Gamma_{31}^2 = 0, \quad \Gamma_{31}^3 = 0, \\ \Gamma_{12}^1 &= 0, \quad \Gamma_{12}^2 = \frac{1}{\theta^1}, \quad \Gamma_{12}^3 = 0, \\ \Gamma_{22}^1 &= -\theta^1, \quad \Gamma_{22}^2 = 0, \quad \Gamma_{22}^3 = 0, \\ \Gamma_{32}^1 &= 0, \quad \Gamma_{32}^2 = 0, \quad \Gamma_{32}^3 = 0, \\ \Gamma_{13}^1 &= 0, \quad \Gamma_{13}^2 = 0, \quad \Gamma_{13}^3 = 0, \\ \Gamma_{23}^1 &= 0, \quad \Gamma_{23}^2 = 0, \quad \Gamma_{23}^3 = 0, \\ \Gamma_{33}^1 &= 0, \quad \Gamma_{33}^2 = 0, \quad \Gamma_{33}^3 = 0. \end{aligned} \quad (15)$$

As can be seen, only the following terms are nonzero

$$\Gamma_{21}^2 = \frac{1}{\theta^1}, \quad \Gamma_{12}^2 = \frac{1}{\theta^1}, \quad \Gamma_{22}^1 = -\theta^1. \quad (16)$$

The Christoffel symbols are symmetric in the two lower indices, that is

$$\Gamma_{ij}^k = \Gamma_{ji}^k. \quad (17)$$

### 3.6 Covariant Differentiation

With the above results in hand it is now possible to discuss covariant differentiation for scalar, vector, and tensor quantities in a curvilinear coordinates system, such as cylindrical coordinates.

The covariant derivative of a scalar field is (using  $d$  index for the derivative index):

$$v_{|d} = v_{,d}, \quad (18)$$

where a vertical bar indicates covariant differentiation with respect to the subscript that follows the vertical bar.

The covariant derivative of a vector is

$$\begin{aligned} v_{|d}^i &= v_{,d}^i + \Gamma_{sd}^i v^s, \\ v_{i|d} &= v_{i,d} - \Gamma_{id}^s v_s. \end{aligned} \quad (19)$$

The covariant derivative of a second order tensor is

$$\begin{aligned} T_{|d}^{ij} &= T_{,d}^{ij} + \Gamma_{dm}^i T^{mj} + \Gamma_{dm}^j T^{im}, \\ T_{ij|d} &= T_{ij,d} - \Gamma_{id}^m T_{mj} - \Gamma_{jd}^m T_{im}, \\ T_{j|d}^i &= T_{j,d}^i + \Gamma_{dm}^i T_j^m - \Gamma_{jd}^m T_m^i. \end{aligned} \quad (20)$$

The covariant derivative of a general tensor is

$$T_{pq|d}^{rs} = T_{pq,d}^{rs} - \Gamma_{pd}^m T_{mq}^{rs} - \Gamma_{qd}^m T_{pm}^{rs} + \Gamma_{dm}^r T_{pq}^{ms} + \Gamma_{dm}^s T_{pq}^{rm}. \quad (21)$$

A fundamental result is that the covariant derivative of any metric tensor is zero.

### 3.7 Recipe for index placement for Christoffel terms in covariant derivative

Suppose we have a tensor with a number of lower and upper indices. We must, in turn, go through each of the indices of the original tensor to generate the Christoffel terms of the covariant derivative. Consider one such index. If it is lower then it is one of the negative Christoffel terms in the covariant derivative, if it is upper then it is one of the positive Christoffel terms in the covariant derivative.

For those terms containing Christoffel symbols in a covariant derivative, follow this procedure:

- (a) Pick the index to consider. Assume the covariant derivative is being taken with respect to the index  $d$ . (Example:  $T_{j|d}^i$ , consider index  $i$ .)
- (b) The position of the considered index  $i$  and  $d$  go in their same positions (up or down) on the Christoffel symbol. (Example continued: put index  $i$  and  $d$  on the Christoffel symbol,  $+\Gamma_{d?}^i$ )
- (c) The summation index  $s$  goes in the remaining position on the Christoffel symbol. (Example continued: place summation index,  $+\Gamma_{ds}^i$ )
- (d) The twin summation index goes in the position of the considered index on the tensor. All other tensor indices go in the same place as they were on the original tensor. (Example continued: place twin summation index and remaining indices on tensor,  $+\Gamma_{ds}^i T_j^s$ , which is the Christoffel term associated with upper index  $i$ )

The covariant derivative discussed in the preceding recipe has two more terms. The complete final form of the example covariant derivative is given below.

$$T_{j|d}^i = T_{j,d}^i + \Gamma_{ds}^i T_j^s - \Gamma_{dj}^s T_s^i. \quad (22)$$

### 3.8 Gradient, Laplacian, Divergence, and Curl in Cylindrical Coordinates

The gradient and the Laplacian are derived in coordinate invariant form using the tensor framework of covariant differentiation [2]. The derivations are provided below.

Use covariant differentiation to get the components of the gradient of a scalar function:

$$\phi_{|i} = \phi_{,i}. \quad (23)$$

In the ensuing derivation, to ensure that summation always occurs between upper and lower indices in general coordinates, it is necessary to have the index raised. Consequently, it is rewritten as

$$\phi^{||i} = g^{ij}\phi_{||j}. \quad (24)$$

To get the Laplacian, the divergence is applied to get

$$\nabla \cdot (\nabla \phi) = \phi^{||i}_{||i} = (g^{ij}\phi_{||j})_{||i}. \quad (25)$$

Noting that the covariant derivative of the metric tensor is zero, the covariant derivative product rule of the r.h.s. results in only one term, that is

$$\nabla \cdot (\nabla \phi) = \phi^{||i}_{||i} = (g^{ij}\phi_{||j})_{||i}. \quad (26)$$

Then, applying the covariant differentiation, the coordinate invariant Laplacian is

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = g^{ij}(\phi_{,ji} - \Gamma_{ji}^k \phi_{,k}). \quad (27)$$

The divergence of a vector

$$\nabla \cdot \mathbf{V} = V_{i||i} = V_{i,i} - \Gamma_{ii}^s V_s. \quad (28)$$

The curl of a vector

$$\nabla \times \mathbf{V} = V_{j||i} \epsilon_{ijk} = (V_{j,i} - \Gamma_{ji}^s V_s) \epsilon_{ijk}. \quad (29)$$

### 3.9 Examples Using the Preceding Results

#### 3.9.1 3D Laplacian of a Scalar Function in Cartesian Coordinates

Using (27) it is noted for Cartesian coordinates that all  $\Gamma_{ij}^k = 0$  and  $g^{ij} = \delta^{ij}$  so that

$$\nabla^2 \phi = \delta^{ij} \phi_{,ji} = \phi^{,i}_{,i}. \quad (30)$$

Consequently, the Laplacian of a scalar function  $\phi$  in Cartesian coordinates is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{x^2} + \frac{\partial^2 \phi}{y^2} + \frac{\partial^2 \phi}{z^2}. \quad (31)$$

#### 3.9.2 3D Laplacian of a Scalar Function in Cylindrical Coordinates

With (27) in hand the Laplacian in cylindrical coordinates is found. Recall the previously found nonzero Christoffel symbols ( $\Gamma_{21}^2, \Gamma_{12}^2, \Gamma_{22}^1$ ), and note that only  $g^{11}$ ,  $g^{22}$ , and  $g^{33}$  are nonzero. Hence, only the following three terms arise in the summation for the Laplacian in cylindrical coordinates:

$$\begin{aligned} g^{11}(\phi_{,11} - \Gamma_{11}^k \phi_{,k}) &= \phi_{,11} \\ g^{22}(\phi_{,22} - \Gamma_{22}^k \phi_{,k}) &= \frac{1}{(\theta^1)^2}(\phi_{,22} + \theta^1 \phi_{,1}) \\ g^{33}(\phi_{,33} - \Gamma_{33}^k \phi_{,k}) &= \phi_{,33} \end{aligned} \quad (32)$$

Finally, the Laplacian of a scalar function in cylindrical coordinates is

$$\nabla^2\phi = \phi_{,11} + \frac{1}{\theta^1}\phi_{,1} + \frac{1}{(\theta^1)^2}\phi_{,22} + \phi_{,33} \quad (33)$$

In more customary notation, the Laplacian of a scalar function in cylindrical coordinates is

$$\nabla^2\phi = \phi_{,rr} + \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta} + \phi_{,zz} \quad (34)$$

### 3.9.3 Laplacian of a Scalar Function in Polar Coordinates

In this case, the 3D Laplacian in cylindrical coordinates degenerates to the Laplacian for polar coordinates if all third, or  $z$ , coordinate terms are neglected. For polar coordinates, the Laplacian is

$$\nabla^2\phi = \phi_{,11} + \frac{1}{\theta^1}\phi_{,1} + \frac{1}{(\theta^1)^2}\phi_{,22}, \quad (35)$$

or

$$\nabla^2\phi = \phi_{,rr} + \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\theta\theta}. \quad (36)$$

## 4 Spherical Coordinates

Cartesian coordinates in terms of spherical coordinates are (see Figure 2)

$$\begin{aligned} x^1 &= \theta^1 \cos \theta^2 \sin \theta^3 \\ x^2 &= \theta^1 \sin \theta^2 \sin \theta^3 \\ x^3 &= \theta^1 \cos \theta^2. \end{aligned} \quad (37)$$

Spherical coordinates in terms of Cartesian coordinates are

$$\begin{aligned} \theta^1 &= \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \\ \theta^2 &= \cos^{-1} \left( \frac{x^3}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}} \right) \\ \theta^3 &= \tan^{-1} \left( \frac{x^2}{x^1} \right). \end{aligned} \quad (38)$$

Finding basis vectors, metric tensors, Christoffel symbols, and other results follows the same procedures illustrated previously for cylindrical coordinates.

## References

- [1] A. E. Green and W. Zerna. *Theoretical Elasticity*. Clarendon Press, Oxford, 2nd edition, 1968.
- [2] OpenAI. ChatGPT (jan 20 version) Response to prompt for derivation of Laplacian in cylindrical coordinates using covariant differentiation, 2026. Available from [Chatgpt.com](https://chatgpt.com).

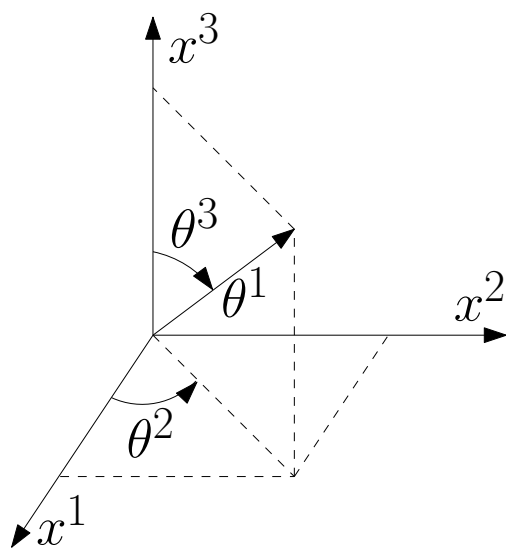


Figure 2: Spherical Coordinates