

Rocket Motors, Incompressibility, and Axisymmetric Problems

Louie L. Yaw

Walla Walla University

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Acknowledgments:

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Outline

1. Motivation: Historical Significance & Presentation of Research
2. Rocket Motors circa 1960
3. Incompressibility
4. Numerical Solution of Elasticity Equations
5. Axisymmetric Research
6. Numerical Implementation
7. Conclusions
8. Questions



1. Motivation: History of Problem of Incompressibility and Presentation of Axisymmetric Research

- The engineering problem of rocket motors circa 1960
- Graduate school professor famous for numerically solving problem of incompressibility – Leonard Herrmann
- My research on axisymmetric problems

***This presentation includes excerpts from recently published paper:
L. L. Yaw, Axisymmetric Virtual Elements for Problems of Elasticity
and Plasticity, *Int. Journal for Num. Methods in Engineering*, 2024.**



Prof. Leonard Herrmann, 1995



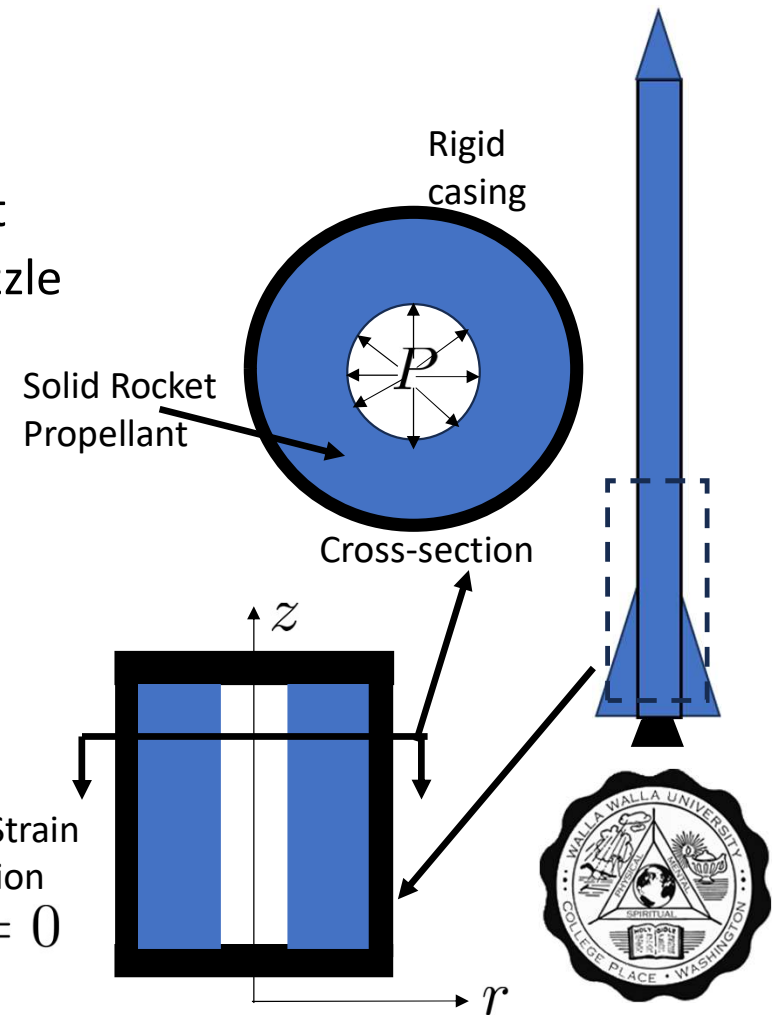
2. The engineering problem of rocket motors circa 1960

- Solid rocket motors, how stress arises
 - Lit by igniter to burn from inside out
 - Gas pressure builds inside the hollow propellant
 - Pressurized gas exhausts through the rocket nozzle
 - Gas pressure causes radial, hoop, and z stresses
 - The loading is axisymmetric



Plane Strain
Nonzero strains are
 $\epsilon_{rr}, \epsilon_{tt} = \epsilon_{\theta\theta}$

Plane Strain
Condition
 $u_z = 0$



2. The engineering problem of rocket motors circa 1960

- Manufacturing solid rocket motors for the space race
 - Aerojet (1964) under contract for largest solid rocket ever built (260-inch)
 - Aerojet involved in the Apollo space program
 - Chemical (solid) Propellant
 - Ammonium perchlorate as oxidizer
 - Aluminum powder as fuel additive
 - A binder to hold the propellant together (polyurethane or polybutadiene acrylonitrile)
 - The binder/propellant composite is hard rubber-like material
 - Resulting material is near incompressible



2. The engineering problem of rocket motors circa 1960

- Rocket motors



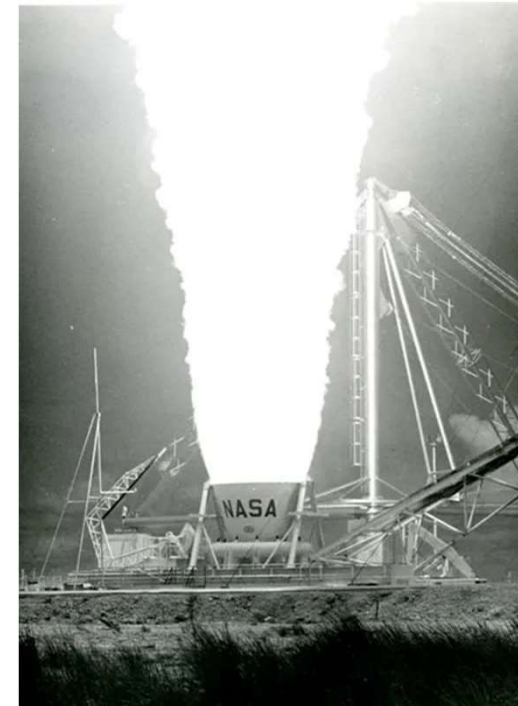
AJ-260X

Aerojet AJ-260X Solid Rocket Motor.

This strap-on booster is based on the AJ-260, the single most powerful rocket engine ever tested. The 260 inch diameter solid rocket motor was so large that it was tested by putting the motor nose-first into a pit and allowing the engine to exhaust vertically.

These boosters would have been powered by burning Polybutadiene acrylonitrile (PBAN) fuel and binding agent with an ammonium perchlorate oxidizer. Intended for use as first stage boosters for uprated Saturn V's, they were also considered as a stand-alone first stage along with an S-IVB upper stage for a Saturn I-B variant called the Saturn INT-05.

- Gross Mass: 1,648,355 kg (3,634,000 lb)
- Dry Mass: 156,126 kg (344,198 lb)
- Propellant Mass: 2,525,000 kg (5,567,000 lb)
- Thrust: 35,391.395 kN (7,956,302 lbf)
- I_{sp} : 263 seconds at sea level
- Burn time: 114 seconds
- Diameter: 6.6 m (260 inches or 21.6 ft)
- Length: 30.48 m (99.99 ft)



Aerojet-General CORPORATION
SOLID ROCKET PLANT . SACRAMENTO, CALIFORNIA



2. The engineering problem of rocket motors circa 1960

- Analyzing Solid rocket motors
 - Grain analysis
 - **Significance of Grain Analysis:** Solid propellant (“grain”) structural analysis is required to ensure it does not crack or fail under load. Dr. Herrmann and his team worked on the structural mechanics of these motors, which were intended to provide reliable and powerful, yet cost-effective propulsion for large-scale space launch vehicles
 - **SIDE NOTE** –recent Artemis II used two solid motor booster rockets, the largest ever made (2026)



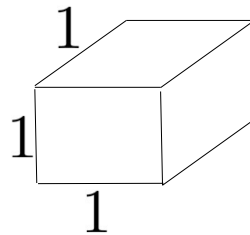
3. Incompressibility

- What does incompressibility mean? No change in volume.
- How is compressibility characterized? Poisson's ratio

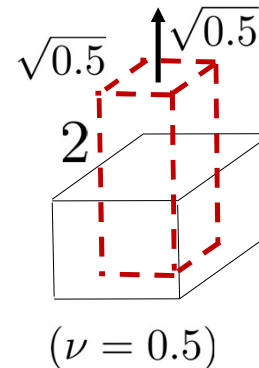
$$\nu = -\frac{\epsilon_{lat}}{\epsilon_{long}}$$

Greater incompressibility

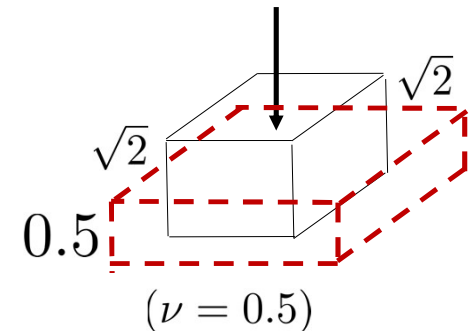
$$0 \leq \nu \leq 0.5$$



Original Volume = $1(1)(1) = 1$



Final Volume = $2\sqrt{0.5}\sqrt{0.5} = 1$



Final Volume = $0.5\sqrt{2}\sqrt{2} = 1$

Numerical = approximate solutions obtained by computer calculations

Numerical problems arise as Poisson's ratio approaches 0.5

$$\nu \rightarrow 0.5$$

Warning !!!!!

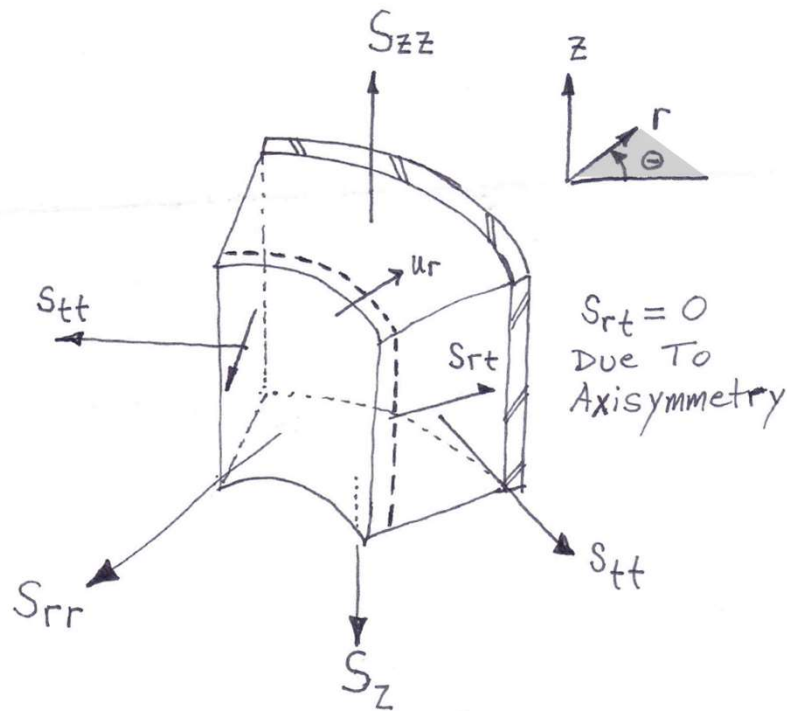
concrete $\rightarrow \nu \approx 0.2$
steel $\rightarrow \nu \approx 0.3$
aluminum $\rightarrow \nu \approx 0.33$
rubber $\rightarrow \nu \approx 0.4999$



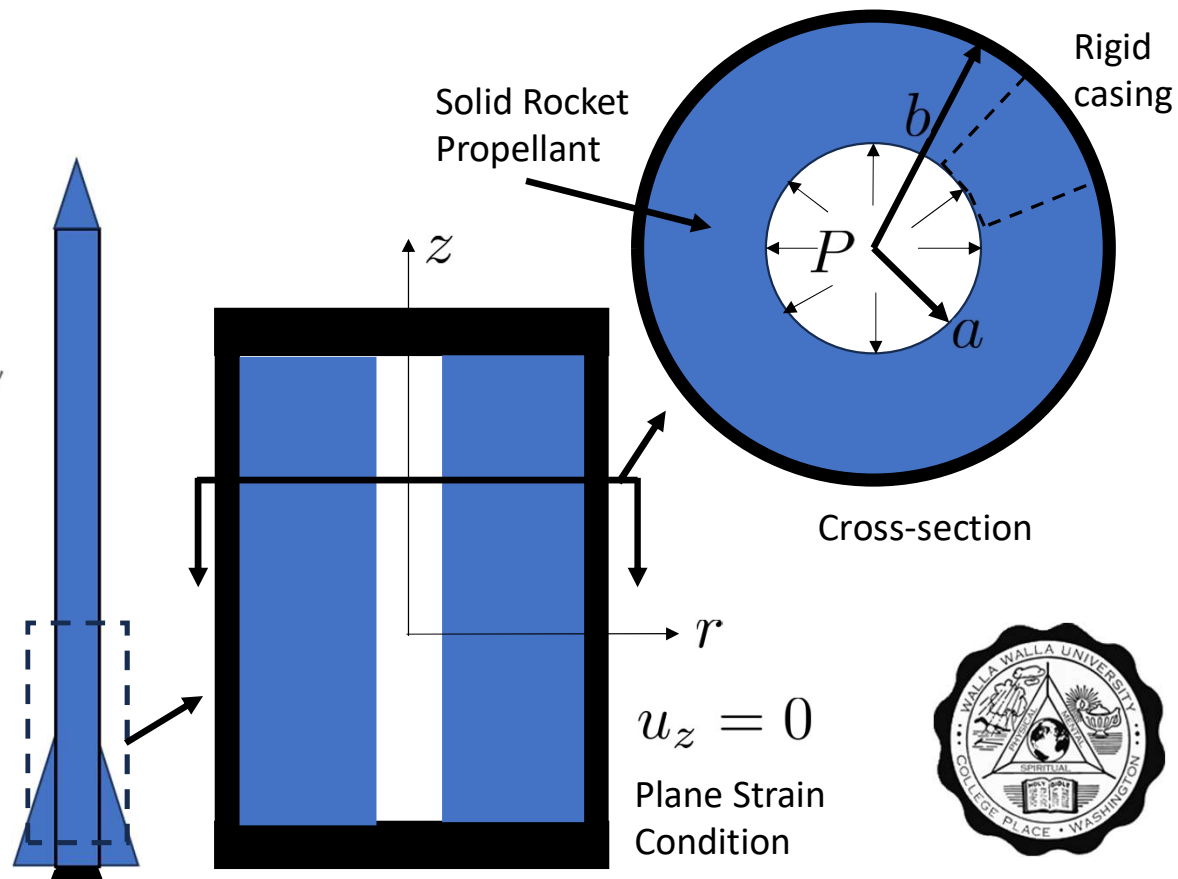
4. Numerical Solution of Linear Elasticity Equations

- Quantities to determine u_r , S_{rr} , S_{zz} , S_{tt}

Axisymmetry about an axis, must be true for loading and geometry



For Ensuing Examples
Internal pressure, $P = 1.0$
 $E = 1$ $\nu = 0.49$



4. Numerical Solution of Linear Elasticity Equations

- Classical Elasticity Equations (15 unknowns, stress, strain, displacements)
 - Stress strain relations (6 equations)

$$\tau_j^i = \lambda \epsilon_k^k \delta_j^i + 2\mu \epsilon_j^i - (3\lambda + 2\mu)\alpha \Delta T \delta_j^i$$

where $\lambda = \frac{2\mu\nu}{1-2\nu}$, *mean strain* $= \epsilon_k^k = \theta$

This presentation
assume $\Delta T = 0$
throughout

- Strain displacement relations (6 equations)

$$\epsilon_j^i = \frac{1}{2}(u^i_{|j} + g^{ik}u_{j|k})$$

$\nu \rightarrow 0.5$

Warning !!!!!

- Equilibrium equations (3 equations)

$$\tau_j^i_{|i} + F_i = 0$$



4. Numerical Solution of Linear Elasticity Equations

- Solving the classical linear elasticity equations for axisymmetric problem
 - Requires the solution of following ODE

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} = 0$$

Where the boundary conditions are

$$\text{at } r = a, \quad S_{rr}(a) = P$$

$$\text{at } r = b, \quad u_r = 0$$

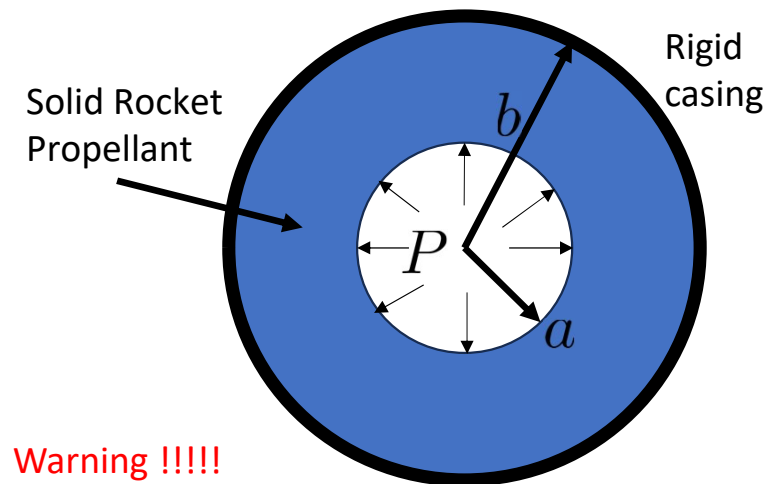
$$\mu = \frac{E}{2(1-\nu)}, \quad \lambda = \frac{2\mu\nu}{1-2\nu}$$

$$\nu \rightarrow 0.5$$

Warning !!!!!

Note:

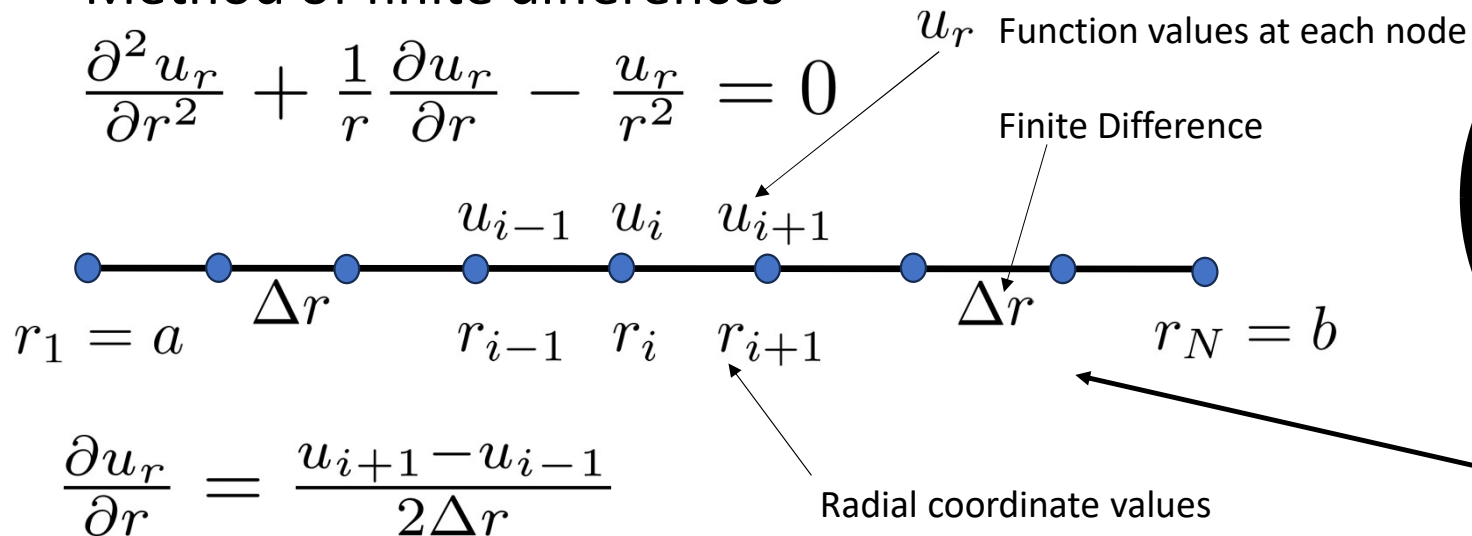
$$S_{rr}(a) = (\lambda + 2\mu) \left(\frac{\partial u_r}{\partial r} + \frac{\nu}{1-\nu} \frac{u_r}{a} \right) = P$$



4. Numerical Solution of Linear Elasticity Equations

- Method of finite differences

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} = 0$$



$$\frac{\partial u_r}{\partial r} = \frac{u_{i+1} - u_{i-1}}{2\Delta r}$$

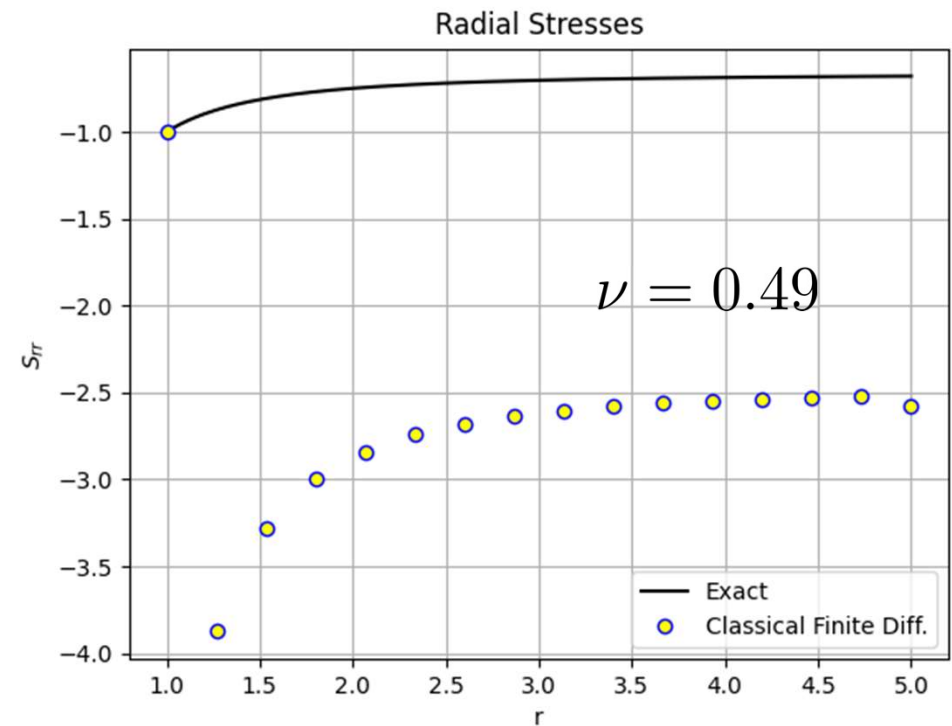
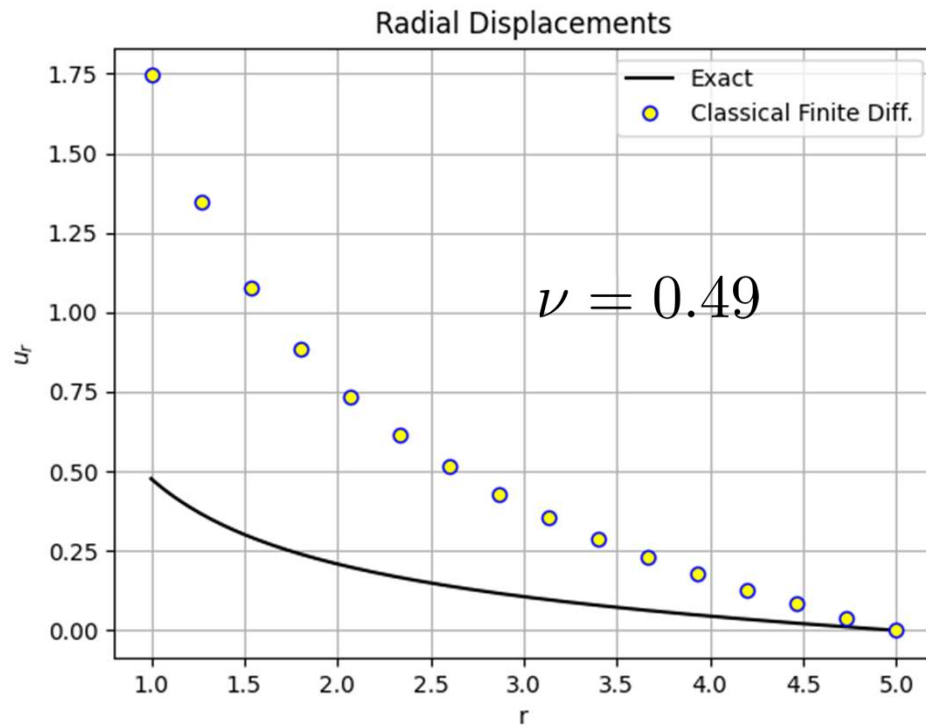
$$\frac{\partial^2 u_r}{\partial r^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2}$$

- Substitute into ODE for each node i , get N equations and N unknowns (with special consideration at boundary nodes)



4. Numerical Solution of Linear Elasticity Equations

- Using classical linear elasticity equations for axisymmetric problem



For small Poisson's ratio the classical approach works.
Yet, as Poisson's ratio approaches 0.5 significant errors result.

16 points



4. Numerical Solution of Reformulated Linear Elasticity Equations

- Reformulated Elasticity Equations (Herrmann et. al. 1964)

- Stress strain relations

$$\tau_j^i = H\delta_j^i + \mu[2\epsilon_j^i - \theta\delta_j^i + \alpha\Delta T\delta_j^i],$$

This presentation
assume $\Delta T = 0$
throughout

where $H = \frac{3P}{2(1+\nu)}$, *mean strain* $= \epsilon_k^k = \theta$,

mean pressure $= P = \frac{1}{3}\tau_k^k$

- Strain displacement relations

$$\epsilon_j^i = \frac{1}{2}(u^i_{|j} + g^{ik}u_{j|k})$$

- Equilibrium equations

$$H_{|i}\delta_j^i + \mu[\nabla^2 u_j + (\alpha\Delta T)_{|i}\delta_j^i] + F_i = 0$$

$$\theta - 3\alpha\Delta T - \left(\frac{1-2\nu}{\mu}\right) H = 0,$$

Herrmann and Toms, "A reformulation of the elastic field equation, in terms of displacements, valid for all admissible values of Poisson's ratio", *ASME Journal of Applied Mechanics*, 1964.



4. Numerical Solution of Reformulated Linear Elasticity Equations

- Solving the reformulated linear elasticity equations for axisymmetric problem
 - Requires the solution of following coupled ODEs

$$\frac{3}{2(1+\nu)} \frac{\partial P}{\partial r} + \mu \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) = 0$$

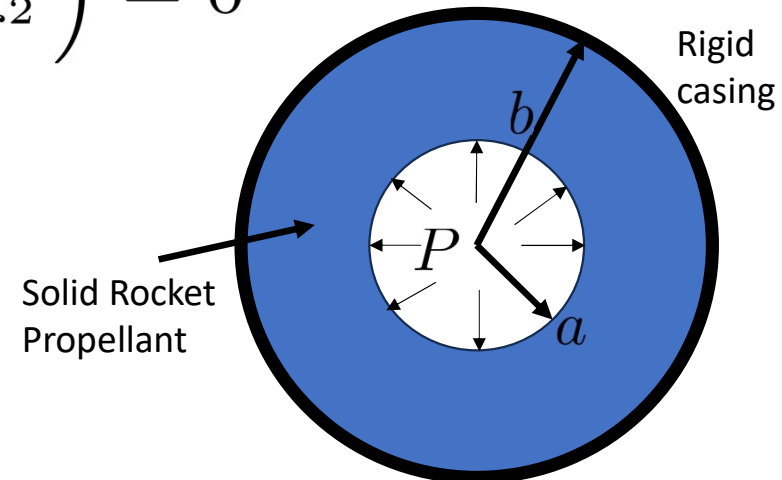
$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} - \frac{3(1-2\nu)}{E} P = 0$$

where the boundary conditions are

$$\text{at } r = a, \quad S_{rr}(a) = P$$

$$\text{at } r = b, \quad u_r = 0$$

Note:
$$S_{rr}(a) = \frac{3P}{2(1+\nu)} + \mu \frac{\partial u_r}{\partial r} - \mu \frac{u_r}{a} = P$$

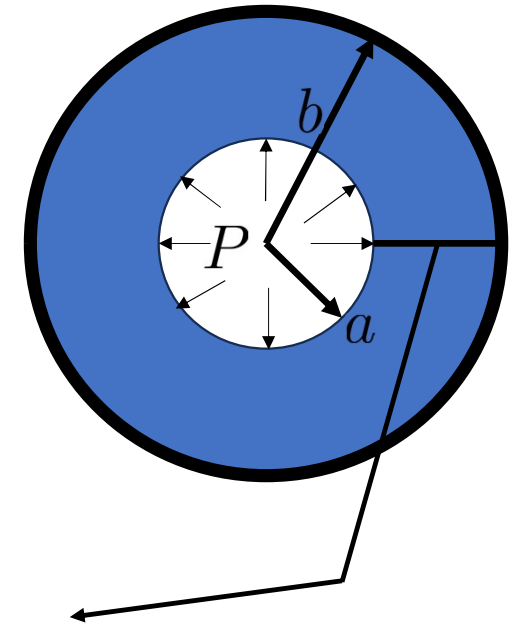
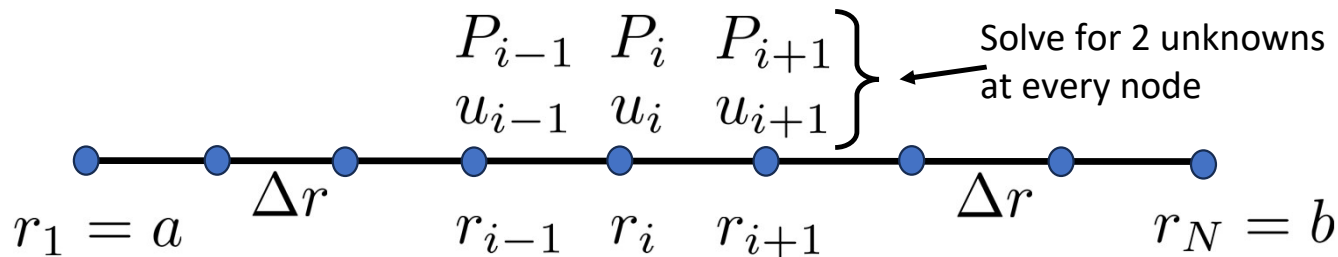


4. Numerical Solution of Reformulated Linear Elasticity Equations

- Method of finite differences

$$\frac{3}{2(1+\nu)} \frac{\partial P}{\partial r} + \mu \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) = 0$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} - \frac{3(1-2\nu)}{E} P = 0$$

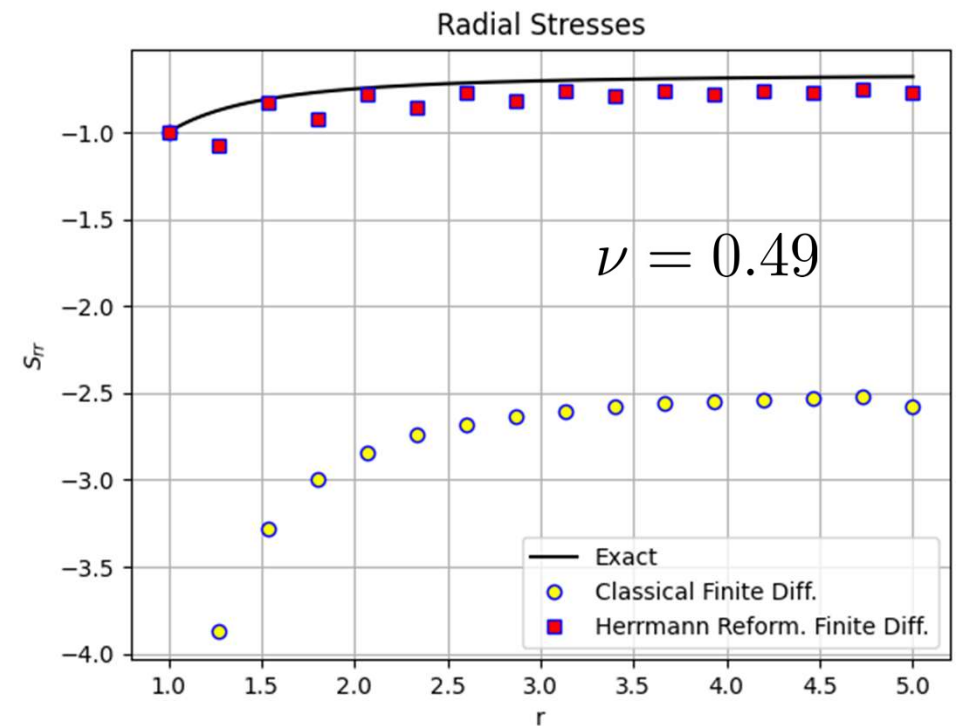
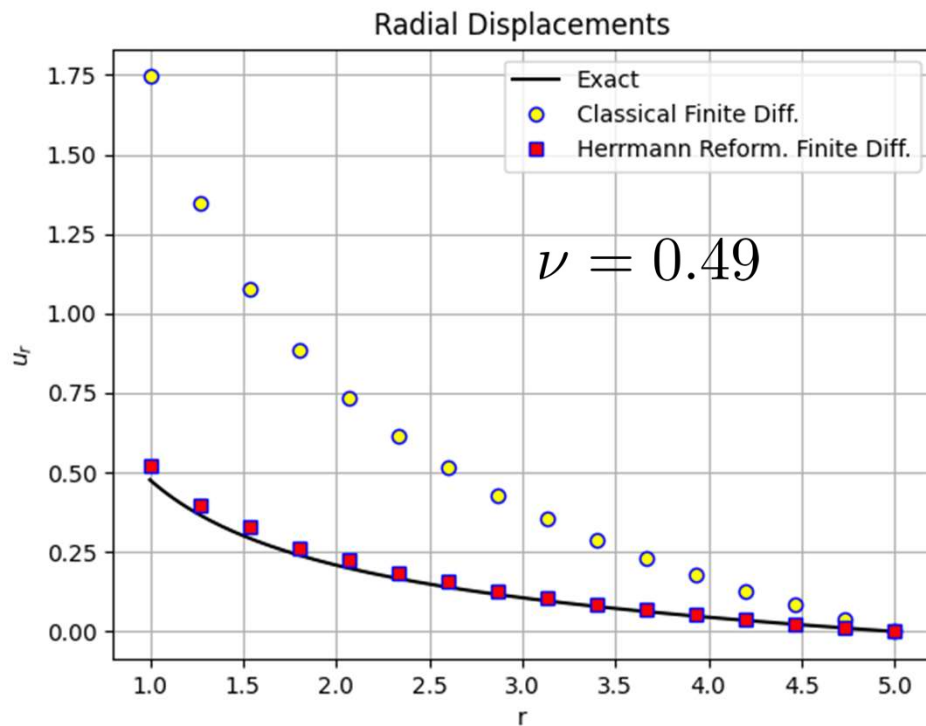


- Substitute into ODE for each node i , get $2N$ equations and $2N$ unknowns (with special consideration at boundary nodes)



4. Numerical Solution of Reformulated Linear Elasticity Equations

- Comparison of Classical and Reformulated Elasticity

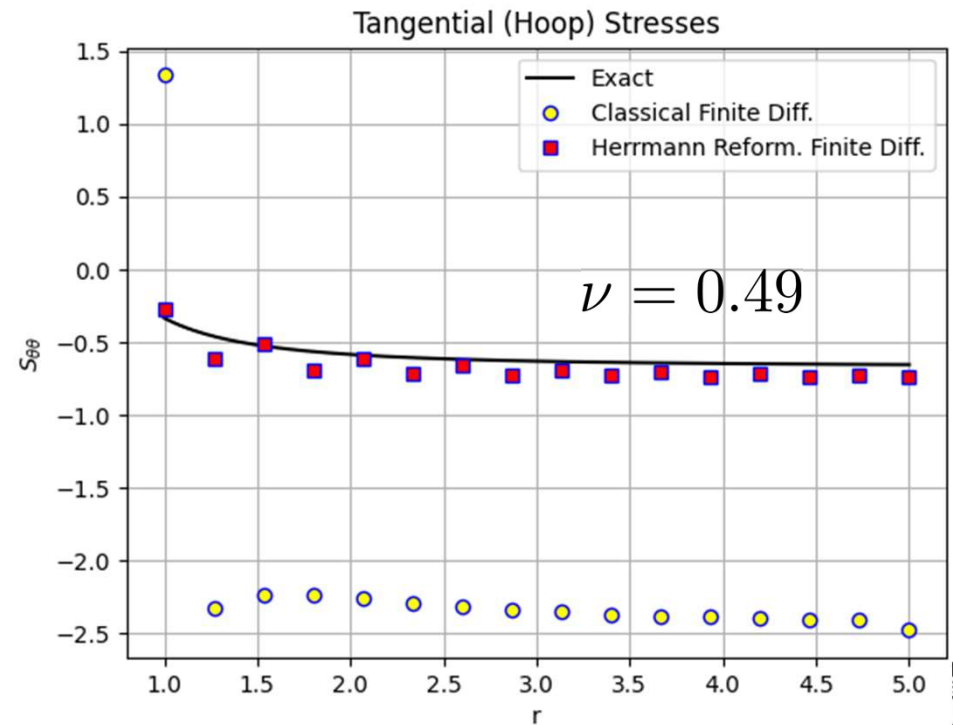
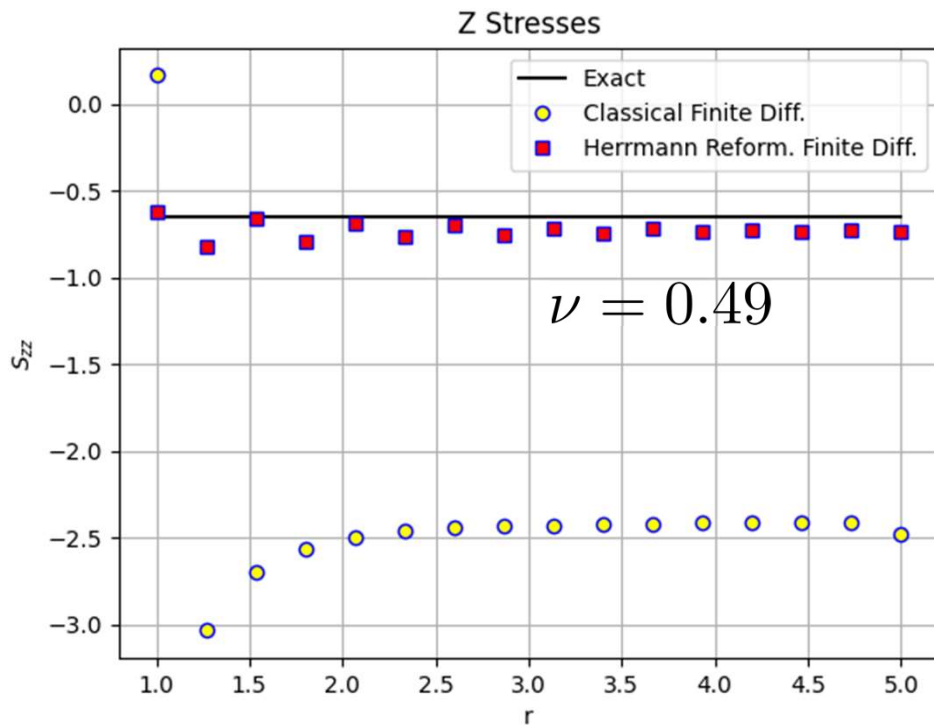


16 points



4. Numerical Solution of Reformulated Linear Elasticity Equations

- Comparison of Classical and Reformulated Elasticity

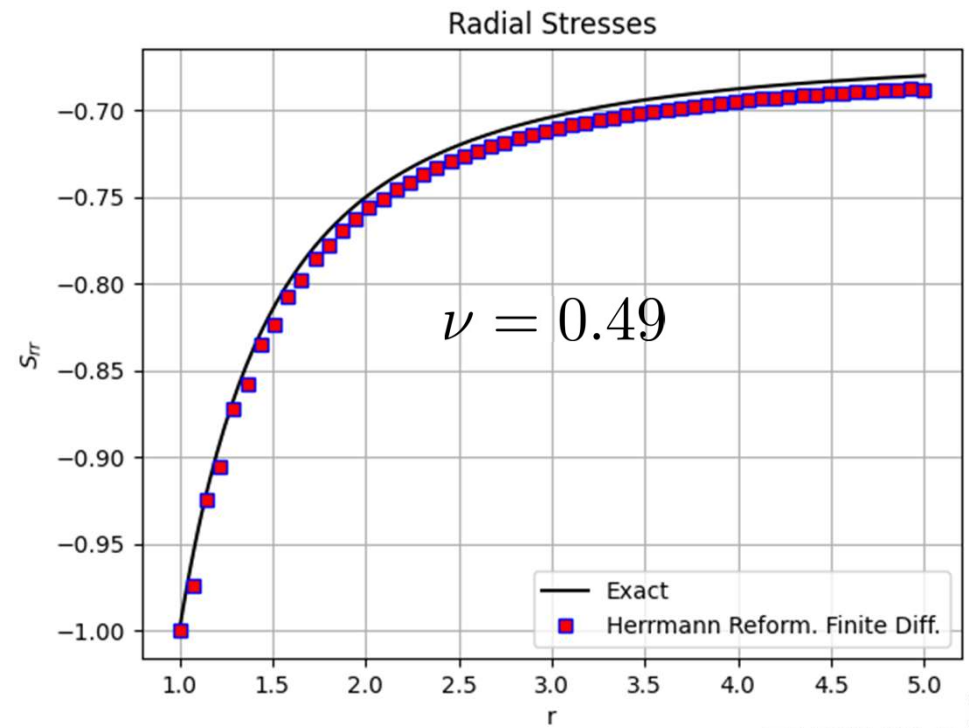
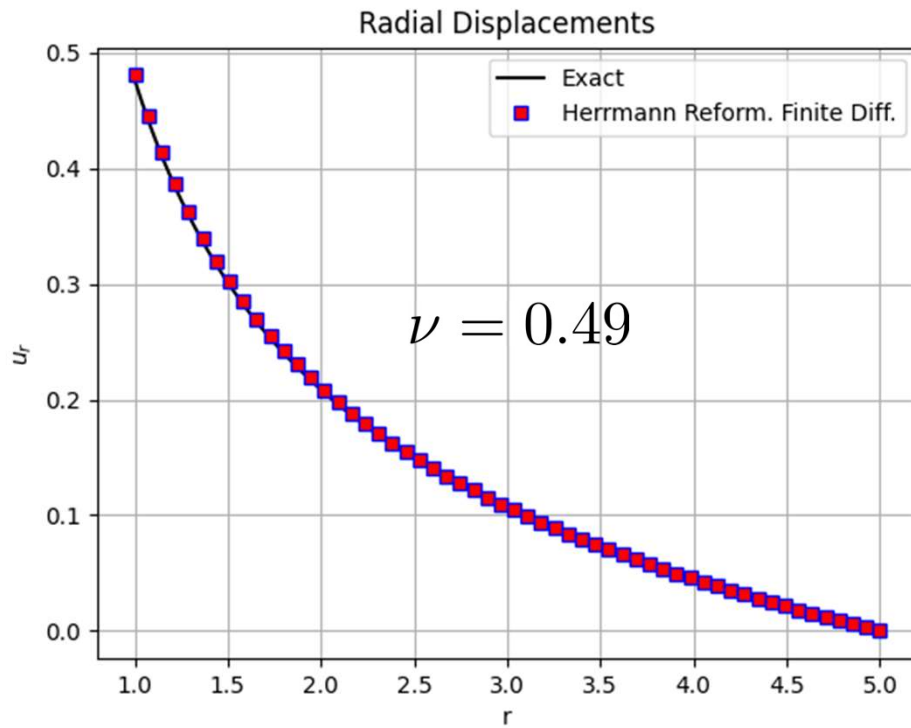


16 points



4. Numerical Solution of Reformulated Linear Elasticity Equations

- Herrmann Reformulated Elasticity With More Points



56 points



5. Axisymmetric Research – Finite & Virtual Elements

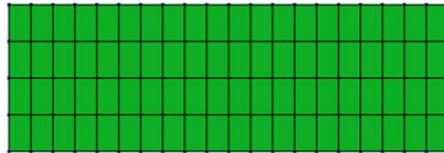
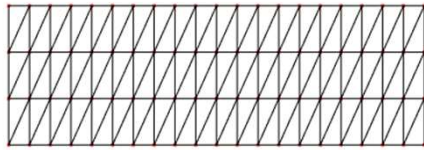
- Finite Element Solution – strategies for near incompressibility
 - pressure displacement solution (Based on Herrmann's idea)
 - selective reduced integration (Hughes, et. al.)
 - mean strain formulation (Nagtegaal, Krysl)
 - method of incompatible modes (Wilson, Taylor, et. al.)
 - B-bar method (Hughes)
 - many others...
- Axisymmetric Virtual Elements (did not exist prior to my research)
 - Special form of stabilization, method used in this presentation (Yaw)

**L. L. Yaw, Axisymmetric virtual elements for problems of elasticity and plasticity,
International Journal for Numerical Methods in Engineering, 2024.**

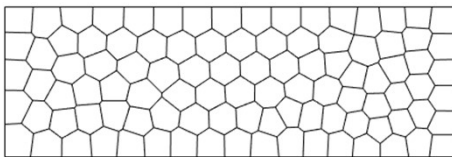


5. Axisymmetric Elements for Elasticity

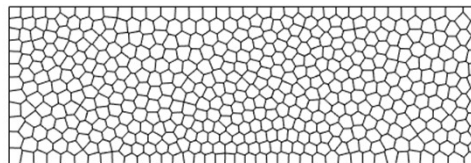
- Example with one quadrilateral section finite or virtual element
- For finite elements, the element sections can be triangular or quadrilateral



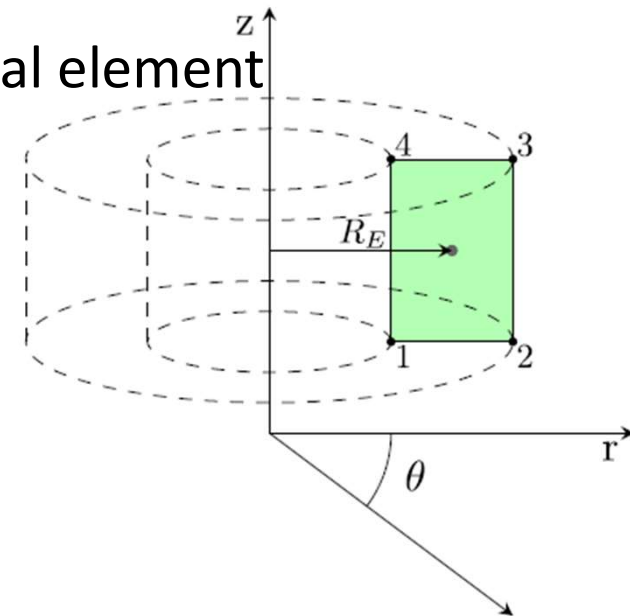
- For virtual elements, the element sections can be arbitrary polygons



100 elements



500 elements

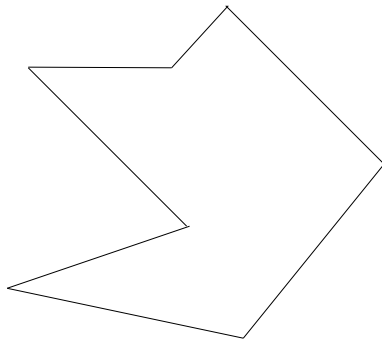


L. L. Yaw, Axisymmetric
Virtual Elements...
IJNME, 2024.

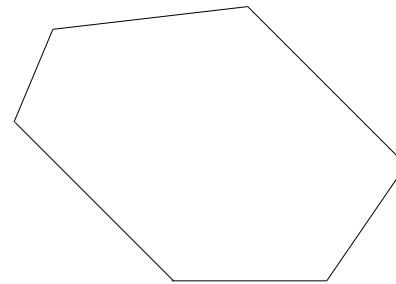


6. Numerical Implementation

- VEM elements can be concave or convex (finite elements cannot be concave)



Concave



Convex



6. Numerical Examples

Lamé Problem

Cylinder Under Internal Pressure

Case of plane strain

$$E_Y = 1000$$

$$\nu = 0.2 \text{ or } 0.49999$$

Inner and outer radii

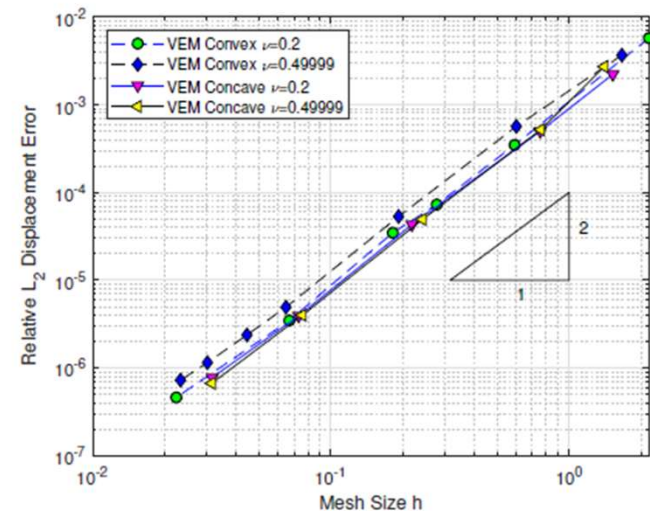
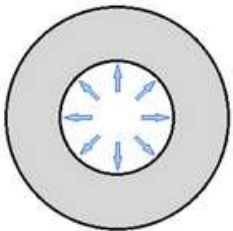
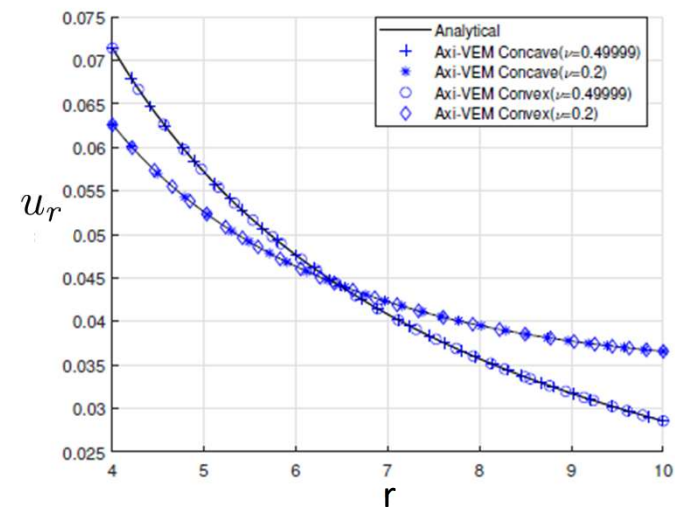
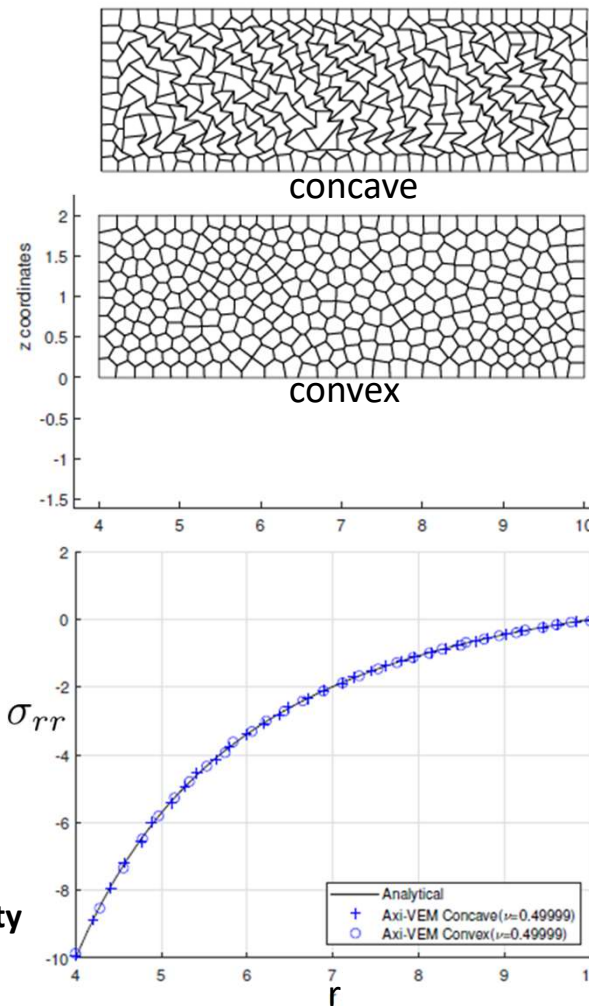
$$a = 4$$

$$b = 10$$

Internal pressure, $p = 10$

Number of V. Elements = 300

L. L. Yaw, Axisymmetric virtual elements for problems of elasticity and plasticity, *IJNME*, 2024.



6. Numerical Examples

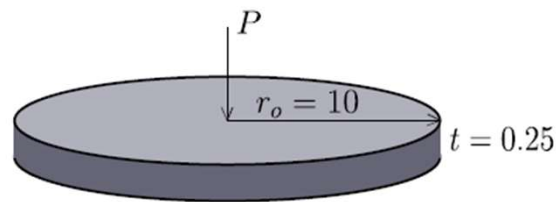
Circular Plate

Case of center point load

Simple support case

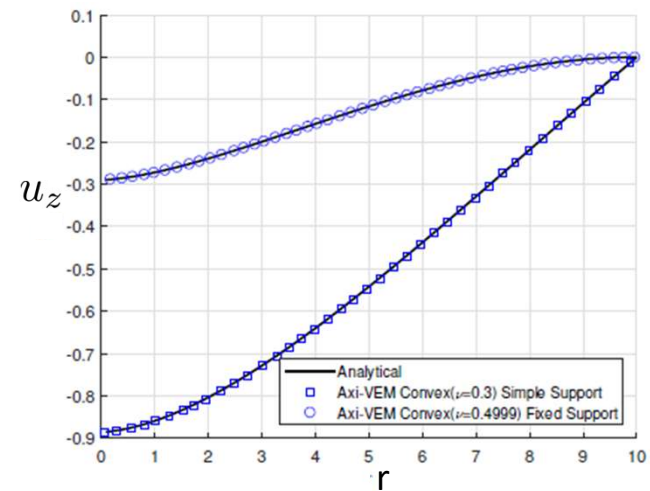
Fixed support case

$$E_Y = 1000$$



	Simple Support		Fixed Support	
ν	Concave	Convex	Concave	Convex
0.0	0.9975	0.9982	0.9847	0.9858
0.3	0.9928	0.9932	0.9751	0.9749
0.49	0.9865	0.9875	0.9638	0.9641
0.499	0.9860	0.9870	0.9627	0.9631
0.49999	0.9863	0.9872	0.9624	0.9624

Circular Plate Normalized Center Deflection Results (9000 elements), $u_z/u_{theoretical}$

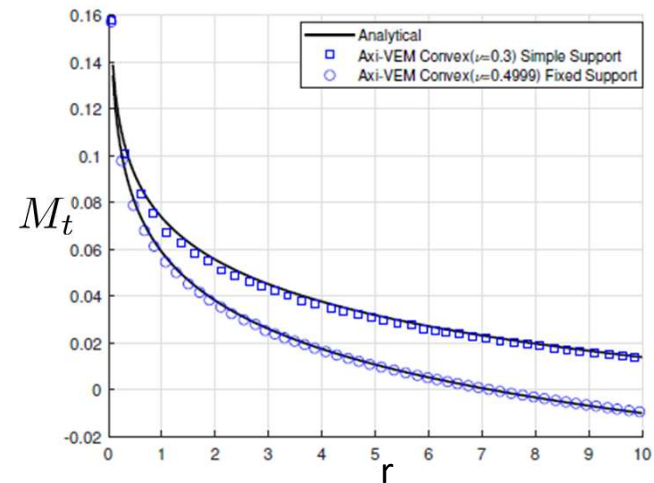
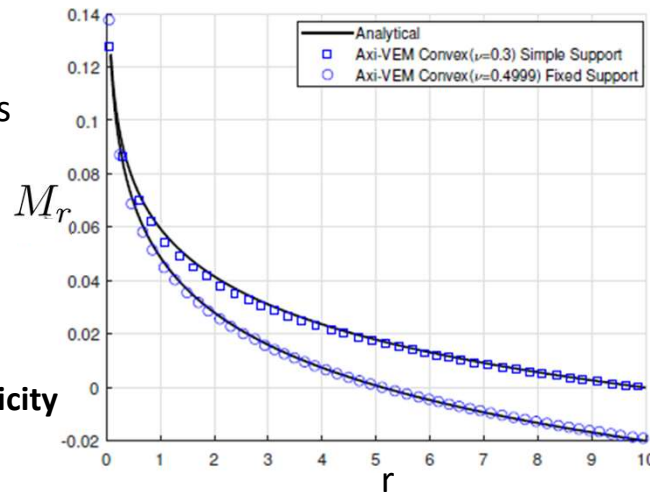


Number of Virtual Elements

For Plots = 18000

$$\nu = 0.3 \text{ or } 0.4999$$

L. L. Yaw, Axisymmetric virtual elements for problems of elasticity and plasticity, *IJNME*, 2024.



6. Numerical Examples

Elastic-Perfectly Plastic Cylinder

Cylinder Under Internal Pressure

Case of plane strain

$$E_Y = 1000$$

$$\sigma_{yield} = 10$$

$$\nu = 0.3$$

Inner and outer radii

$$a = 4$$

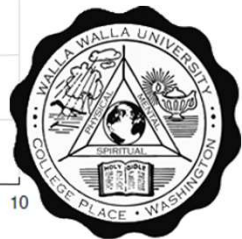
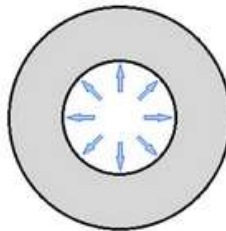
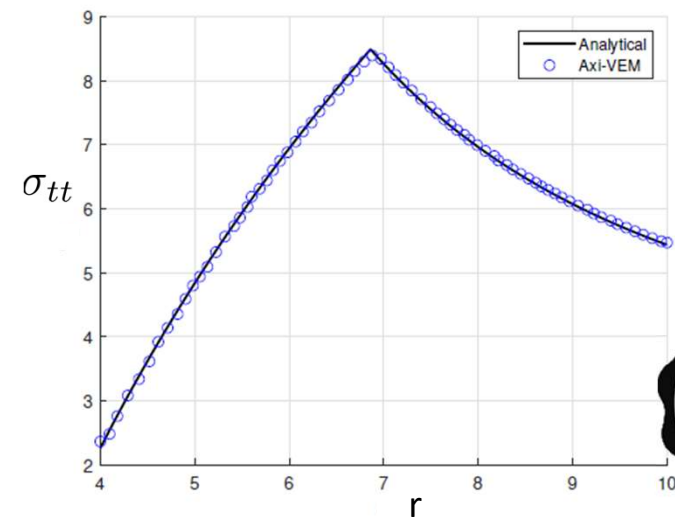
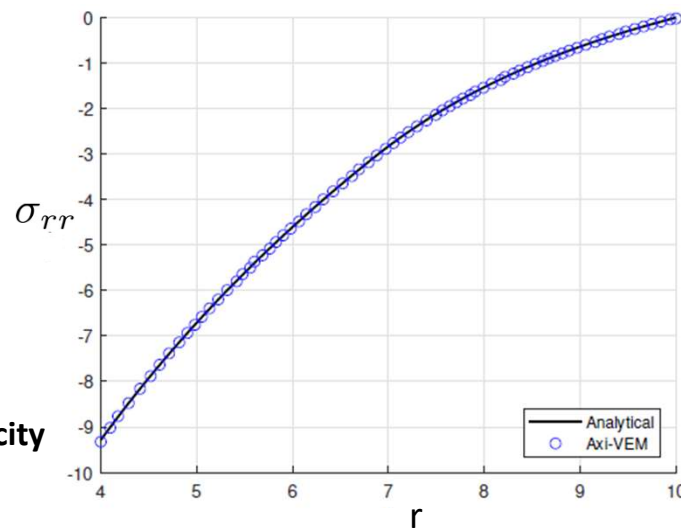
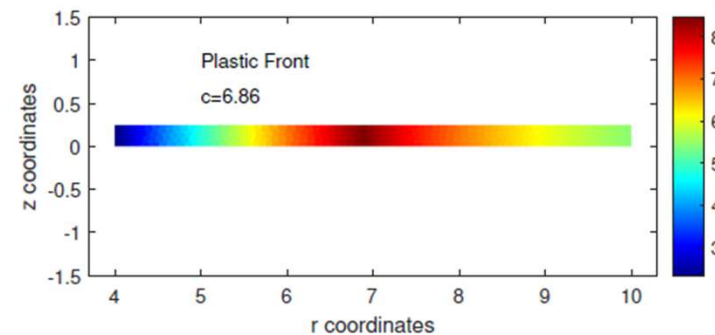
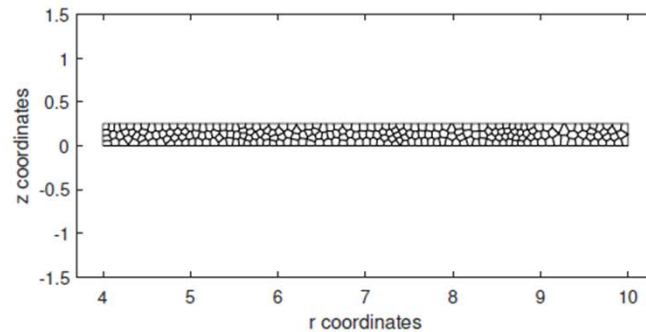
$$b = 10$$

Internal pressure, $p = 9.29$

Number of convex

Virtual Elements = 250

L. L. Yaw, Axisymmetric virtual elements for problems of elasticity and plasticity, *IJNME*, 2024.



6. Numerical Examples

Elastic Perfectly
Plastic Sphere Under
Internal Pressure

$$E_Y = 1000$$

$$\sigma_{yield} = 10$$

$$\nu = 0.3$$

Inner and outer radii

$$a = 4$$

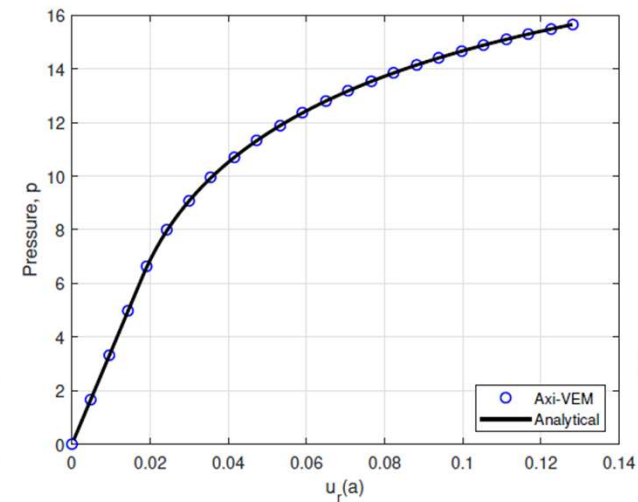
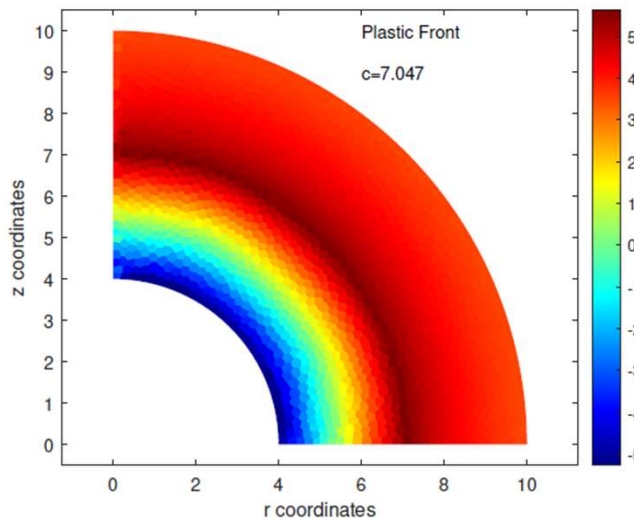
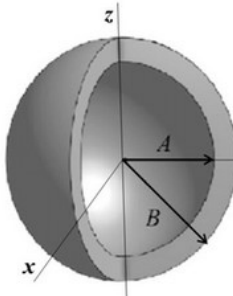
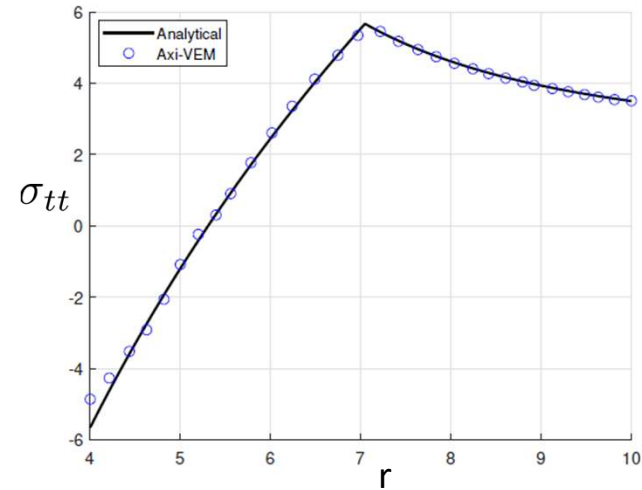
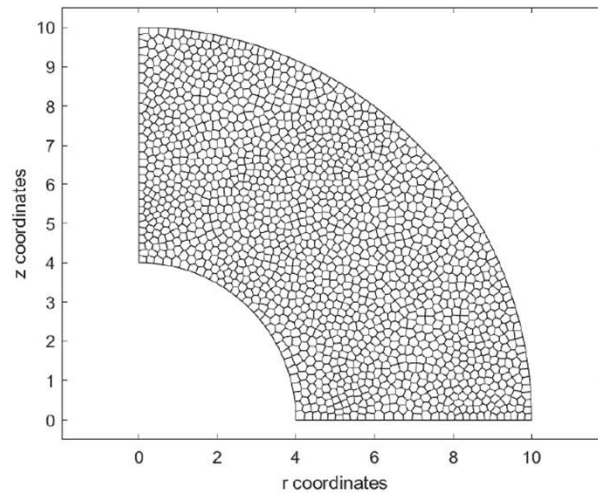
$$b = 10$$

Internal pressure, $p = 15.66$

Number of convex

Virtual Elements = 1600

L. L. Yaw, Axisymmetric virtual
elements for problems of elasticity
and plasticity, *IJNME*, 2024.



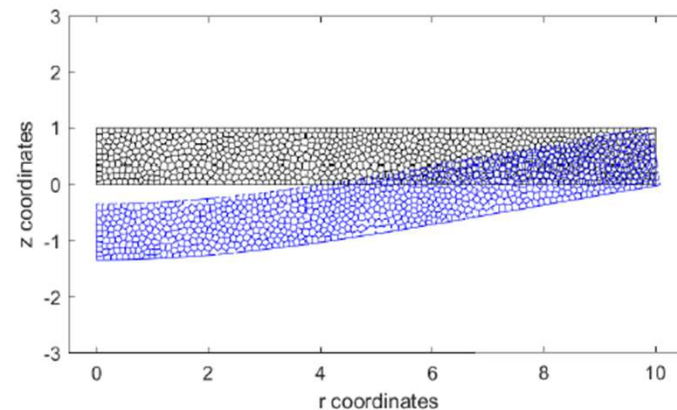
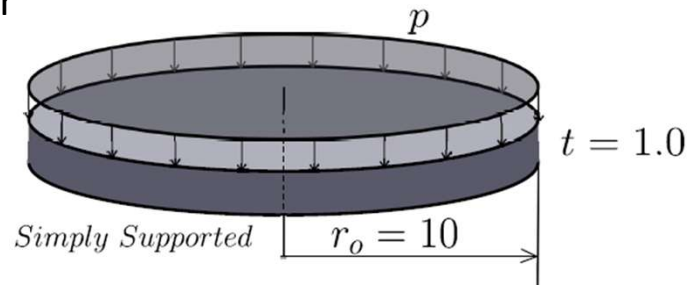
6. Numerical Examples

Elastic Perfectly Plastic
Circular Plate Under
Uniform Pressure

$$E_Y = 10000$$

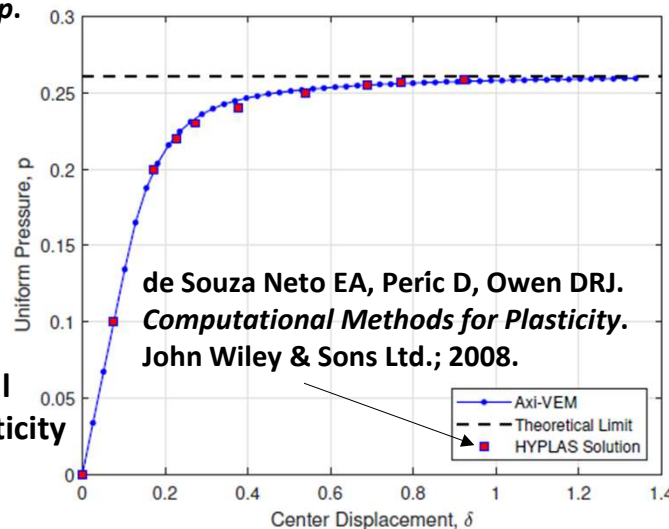
$$\nu = 0.24$$

$$\sigma_{yield} = 16$$

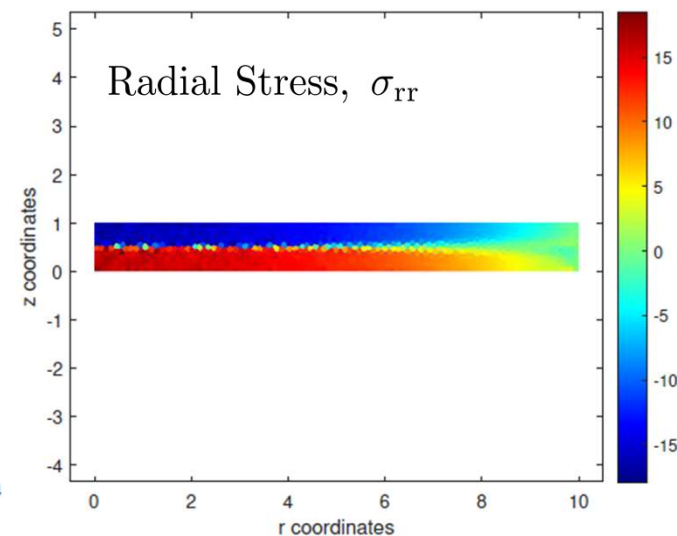


Theoretical Limit Pressure,
 $p=0.2609$, (*Plasticity and Creep*.
CRC Press; 1993)

Axi-VEM Limit Pressure,
 $p=0.2594$



L. L. Yaw, Axisymmetric virtual
elements for problems of elasticity
and plasticity, *IJNME*, 2024.



7. Conclusions

- Successfully solve known problems to confidently solve unknown problems
- Learn by reproducing the results of others
- Axisymmetric virtual elements are viable for problems of elasticity and plasticity
- Verified with representative theoretical & benchmark problems



8. Questions?

